"There are five fundamental operations in mathematics: addition, subtraction, multiplication, division and modular forms."

Definition 0.1. The upper half plane H is the set of all complex numbers with positive imaginary part:

$$H := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Definition 0.2. The special linear group $\mathrm{SL}_2(\mathbb{R})$

$$\mathrm{SL}_2(\mathbb{R}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \text{s.t. } ad - bc = 1\}.$$

We define a group action on $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the Möbius transformation. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \tilde{\mathbb{C}}$, then $gz = \frac{az+b}{cz+d}$. Note that $\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$, so H is stable under the action of $\mathrm{SL}_2(\mathbb{R})$. Also, we find that $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts trivially on H, since for $z \in H$, we have $-1 \cdot z = z$. Hence, we consider the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$.

Definition 0.3. $G := \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ is called the modular group, it is the image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{R})$.

We will have a look at two elements of G, namely

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have,

$$Sz = -\frac{1}{z},\ S^2 = 1,\ Tz = z + 1,\ (ST)^3 = 1.$$

Denote by D the subset of H formed by all points $z \in H$ s.t. $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq \frac{1}{2}$. Below, there is a picture that shows all possible transofrmations of D by the elements

$$\{1, T, TS, ST^{-1}S, S, ST, STS, T^{-1}S, T^{-1}\} \text{ of } G.$$

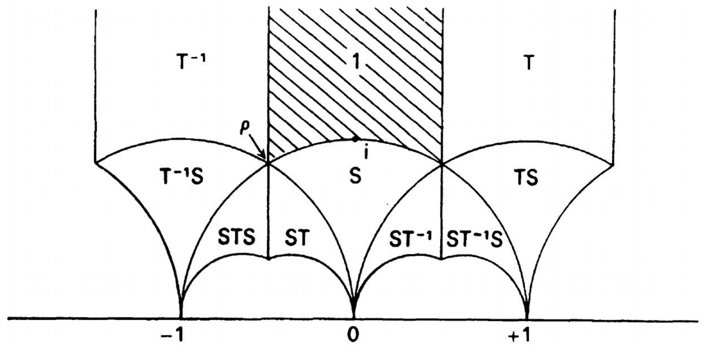

Figure 1: Fundamental Domain

We want to show that D is a fundamental domain for the action of G on H. For this we use the following Theorem:

Theorem 0.4.

1. For every $z \in H, \exists g \in G$ s.t. $gz \in D$

2. Assume that two distinct points $z, z' \in D$ are congruent modulo G. Then, $Re(z) = \pm\frac{1}{2}, z = z' \pm 1$, or $|z| = 1$ and $z' = -\frac{1}{z}$.

3. Let $z \in D$ and let $I(z) = \{g \mid g \in G, gz = z\}$ the stabilizer of z in G. One has $I(z) = \{1\}$ except in the following three cases:

   - $z = i$, then $I(z)$ is the group of order 2 generated by S;
   - $z = \rho = e^{2\pi i/3}$, then $I(z)$ is the group of order 3 generated by ST;
   - $z = -\bar{\rho} = e^{\pi i/3}$, then $I(z)$ is the group of order 3 generated by TS;

Theorem 0.5. The group G is generated by S and T.

Definition 0.6. Let k be an integer. We say that a function f is weakly modular of weight $2k$ if f is meromorphic on H and verifies the relation

$$f(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z).$$

Let g be the image in G of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have:

$$\frac{d(gz)}{dz} = (cz+d)^{-2}$$

Now, we can rewrite the above as:

$$\frac{f(gz)}{f(z)} = (cz+d)^{2k} = (\frac{d(gz)}{dz})^{-k} \iff f(gz)d(gz)^k = f(z)dz^k,$$

which is equivalent to saying that the differential form of weight k, $f(z)dz^k$, is invariant under G. Since G is generated by S and T, we check the invariance by S and T.

Proposition 0.7. Let f be meromorphic on H. The function f is a weakly modular function of weight $2k$ iff it satisfies the two relations:

- $f(z+1) = f(z)$
- $f(-\frac{1}{z}) = z^{2k} f(z)$

Now, since $f(z+1) = f(z)$, f is periodic with period 1. So, we express f as a function of $q = e^{2\pi i z}$ denoted by $\tilde{f}$. $\tilde{f}$ is meromorphic in the disk $|q| < 1$ with the origin removed. If $\tilde{f}$ extends to a meromorphic (resp. holomorphic) function at the origin, we say that f is meromorphic (resp. holomorphic) at infinity. So, $\tilde{f}$ admits a Laurent expansion in a neighbourhood of the origin

$$\tilde{f}(q) = \sum_{-\infty}^{\infty} a_n q^n,$$

where the a_n are zero for n small enough (resp. for $n < 0$).

Definition 0.8. A weakly modular function is called modular if it is meromorphic at infinity.

Definition 0.9. A modular function which is holomorphic everywhere, including infinity, is called a modular form; if such a function is zero at infinity, it is called a cusp form.

We can also define modular forms in terms opf lattices. Recall the following definition of a lattice:

Definition 0.10. A lattice in a real vector space V of finite dimension is a subgroup Γ of V verifying one of the following equivalent conditions:

- Γ is discrete and V/Γ is compact;
- Γ is discrete and generates the $\mathbb{R}$ vector space V;
- There exists an $\mathbb{R}$-basis $(e_1, \dots, e_n)$ of V which is a $\mathbb{Z}$-basis of Γ (i.e. $\Gamma = \mathbb{Z}e_1 \bigoplus \dots \bigoplus \mathbb{Z}e_n$).

Let R be the set of lattices of $\mathbb{C}$ considered as an $\mathbb{R}$ vector space. Let M be the set of pairs (w_1, w_2) of elements of $\mathbb{C}^*$ s.t. $\mathrm{Im}(\frac{w_1}{w_2}) > 0$. We associate the lattice $\Gamma(w_1, w_2) = \mathbb{Z}w_1 \bigoplus \mathbb{Z}w_2$ with basis $\{w_1, w_2\}$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), (w_1, w_2) \in M$. Put $w_1' = aw_1 + bw_2$ and $w_2' = cw_1 + dw_2$. Then, $\{w_1', w_2'\}$ is a basis of $\Gamma(w_1, w_2)$. Setting $z = \frac{w_1}{w_2}, z' = \frac{w_1'}{w_2'}$, gives us $z' = \frac{az+b}{cz+d} = gz$, so $\mathrm{Im}(z') > 0$ and $(w_1', w_2') \in M$. Therefore, we can identify R of $\mathbb{C}$ with the quotient of M by the action of $\mathrm{SL}_2(\mathbb{Z})$. $\mathbb{C}^*$ acts on R by:

$\Gamma \to \lambda\Gamma; (w_1, w_2) \mapsto (\lambda w_1, \lambda w_2), \lambda \in \mathbb{C}^*$. The quotient $M/\mathbb{C}^*$ is identified with H by $(w_1, w_2) \mapsto z = \frac{w_1}{w_2}$, this identification transforms the action of $\mathrm{SL}_2(\mathbb{Z})$ on M into that of $G = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ on H. We can do the same procedure again for modular functions. Let F be a function on R with complex values, $k \in \mathbb{Z}$. F is of weight $2k$ if $F(\lambda\Gamma) = \lambda^{-2k}F(\Gamma)$ for all lattices Γ and all $\lambda \in \mathbb{C}^*$. If $(w_1, w_2) \in M$, we have $F(w_1, w_2)$ as the value of F on the lattice $\Gamma(w_1, w_2)$. So, $F(\lambda w_1, \lambda w_2) = \lambda^{-2k}F(w_1, w_2)$. $F(w_1, w_2)$ is invariant by the action of $\mathrm{SL}_2(\mathbb{Z})$. Since $w_2^{2k}F(w_1, w_2) = F(\frac{w_1}{w_2}, 1)$, which only depends on $z = \frac{w_1}{w_2}$, $\exists f$ on H s.t. $F(w_1, w_2) = w_2^{-2k}f(\frac{w_1}{w_2})$. As F is invariant by $\mathrm{SL}_2(\mathbb{Z})$, f satisfies

$$f(z) = (cz + d)^{-2k} f(\frac{az + b}{cz + d}).$$

Conversely, going from here to a function F on R works too. Hence, we can identify modular functions of weight $2k$ with some lattice functions of weight $2k$.At this point we want to mention that one can associate to a lattice Γ of $\mathbb{C}$ a corresponding elliptic curve $E_\Gamma = \mathbb{C}/\Gamma$. This shows how versatile modular forms can be.

Now we will come to an example of modular functions, called the Eisenstein series.

Let k be an integer greater than 1. If Γ is a lattice of $\mathbb{C}$, put

$$G_k(\Gamma) = \sum_{\gamma \in \Gamma, \gamma \neq 0} \frac{1}{\gamma^{2k}}.$$

This series is convergent and of weight $2k$. We call this the Eisenstein series. We can view G_k as a function of M by

$$G_k(w_1, w_2) = \sum_{m,n,(m,n) \neq (0,0)} \frac{1}{(mw_1 + nw_2)^{2k}}.$$

We get

$$G_k(z) = \sum_{m,n,(m,n) \neq (0,0)} \frac{1}{(mz + n)^{2k}}.$$

Proposition 0.11. Let k be an integer greater than 1. The Eisenstein series $G_k(z)$ is a modular form of weight $2k$. We have $G_k(\infty) = \zeta(2k)$, where ζ denotes the Riemann zeta function.

Proof. $G_k(z)$ is weakly modular of weight $2k$, so we only need to show that it is holomorphic (including at infinity). Suppose $z \in D$. Then,

$$|mz+n|^2 = m^2|z|^2 + 2\,\mathrm{Re}(z)mn + n^2 \geq m^2 - |mn| + n^2 = \frac{m^2 + n^2}{2} + \frac{(|m| - |n|)^2}{2}$$

Since the corresponding sum is convergent, $G_k(z)$ converges normally in D, so, applying this to $G_k(g^{-1}z), g \in G$, it also converges in each of the transforms gD of D by G. We want to prove that G_k has a limit for $\mathrm{Im}(z) \to \infty$. Suppose that z remains in D, due to uniform convergence in D, we can look at the limit term by term, where we get zero, when $m \neq 0$ and $\frac{1}{n^{2k}}$ if $m = 0$. So,

$$\lim_{\mathrm{Im}(z)\to\infty} G_k(z) = \sum_{n\neq 0} \frac{1}{n^{2k}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$$

□

So, the Eisenstein series of lowest weights are G_2 and G_3, of weight 4 and 6 respectively. Let $g_2 = 60G_2, g_3 = 140G_3$. We have $g_2(\infty) = 120\zeta(4)$ and $g_3(\infty) = 280\zeta(6)$, which is $g_2(\infty) = \frac{4}{3}\pi^4, g_3(\infty) = \frac{8}{27}\pi^6$, so putting $\Delta = g_2^3 - 27g_3^2$ gives uf $\Delta(\infty) = 0$ and Δ is a cusp form of weight 12. We call Δ the modular discriminant function.Let $\sigma_k(n)$ be the sum $\sum_{d|n} d^k$ of kth-powers of positive divisors of n. Then, for every integer $k \geq 2$, one gets

$$G_k(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n.$$

This shows us the representation of the Eisenstein series as a Fourier row. This leads us to a different representation of the modular discriminant function.

$$\Delta = g_2^3 - 27g_3^2 = (2\pi)^{12}(q - 24q^2 + 252q^3 - 1472q^4 + \ldots),$$

which is equivalent to

$$\Delta = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Denote by $\tau(n)$ the nth coefficient of the cusp form $F(z) = (2\pi)^{-12}\Delta(z)$. This gives us

$$\sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

The function $n \mapsto \tau(n)$ is called the Ramanujan function. As an example, we have $\tau(2) = -24, \tau(3) = 252, \tau(6) = -6048$. An interesting property of the Ramanujan function is the following product formula:

$$\tau(nm) = \tau(n)\tau(m) \text{ if } (n,m) = 1$$

Note that $\tau(2)\tau(3) = -6048 = \tau(6)$. In addition to the product formula, the Ramanujan function also has the following upper bound $\tau(n) = \mathcal{O}(n^6)$.

Let f be a meromorphic function on H, not identically zero, and let p be a point of H. The integer n s.t. $\frac{f}{(z-p)^n}$ is holomorphic and non-zero at p is called the order of f at p and is denoted by $v_p(f)$. When f is a modular function of weight $2k$, $v_p(f) = v_{g(p)}(f)$ if $g \in G$, so $v_p(f)$ depends only on the image of p in H/G. We define $v_\infty(f)$ as the order for $q = 0$ of $\tilde{f}(q)$. Denote by e_p the order of the stabilizer of the point p. We have $e_p = 2$ (resp. $e_p = 3$) if p is congruent modulo G to i (resp. to ρ) and $e_p = 1$ otherwise.

Theorem 0.12. Let f be a modular function of weight $2k$, not identically zero. One has:

$$v_\infty(f) + \sum_{p \in H/G} \frac{1}{e_p} v_p(f) = \frac{k}{6}$$

In order to propve this theorem, one needs to integrate $\frac{1}{2i\pi}\frac{df}{f}$ on the boundary of D, which needs tools from complex analysis.

Definition 0.13. If k is an integer, we denote by M_k (resp. M_k^0) the $\mathbb{C}$ vector space of modular forms of weight $2k$ (resp. of cusp forms of weight $2k$)

Theorem 0.14.

- We have $M_k = 0$ for $k < 0$ and $k = 1$.
- For $k = 0, 2, 3, 4, 5$, M_k is a vector space of dimension 1 with basis $1, G_2, G_3, G_4, G_5$; we have $M_k^0 = 0$.
- Multiplication by Δ defines an isomorphism of M_{k-6} onto M_k^0.

This theorem can be proven by the valence formula given above.

Corollary 0.15. We have

$$dim(M_k) = \begin{cases} [\frac{k}{6}], k \equiv 1 \text{ if } (\text{mod } 6), k \geq 0 \\ [\frac{k}{6}] + 1, k \not\equiv 1 \text{ if } (\text{mod } 6), k \geq 0 \end{cases}$$

We have that

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_k \pi^{2k},$$

where the B_k denote the Bernoulli numbers, defined by the power series expansion:

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_k \frac{x^2 k}{(2k)!}.$$

Using these numbers, we define now

$$\gamma_k = (-1)^k \frac{4k}{B_k},$$

which leads us to the following representation of the Eisenstein series:

$$G_k(z) = 2\zeta(2k) E_{2k}(z),$$

where

$$E_{2k}(z) = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

All of this leads us the the following Proposition:

Proposition 0.16. The ring $M_*(\Gamma)$ is freely generated by the modular forms E_4 and E_6

The general idea of this proof is to show that the modular forms $E_4(z)$ and $E_6(z)$ are algebraically independent. If E_4 and E_6 were algebraically dependent, then one can show that the forms E_4^3 and E_6^2 of weight 12 must be proportional. However, a direct calculation shows that they are not proportional, thus E_4 and E_6 need to be algebraically independent. Using then the dimension formula above gives us the proof.