Spherical Codes and Designs

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1 Introduction

In the following we want to study "good" finite subsets of the unit sphere S^{n-1}. We are going to look at two different types of set, namely spherical codes and spherical designs.

The idea of spherical codes is to have the points on the unit sphere to be as far away as possible from each other, i.e. the minimum distance should be large. In contrast to this, the idea of spherical designs is to approximate the sphere globally well.

The main references for these notes are [DGS77] and [BB09].

We will give the precise definitions and some examples in section 2. In section 3, we are then going to introduce a class of special polynomials that are needed later to prove bounds on the size of spherical codes and designs in section 4. Finally, we will look at tight designs and their classification in section 5.

2 Definitions and Examples

First, we will now look at spherical codes.

Definition. Let $A \subseteq [-1, 1)$. Then $X \subseteq S^{n-1}$ is a spherical A-code if for all $x \neq y \in X$ we have $\langle x, y\rangle \in A$.

Above, we wanted the minimum distance to be large, but in the definition we are talking about scalar product, however they are related through the following remark:

Remark. For any $x, y \in S^{n-1}$ we have $d(x, y) = \sqrt{2(1 - \langle x, y\rangle)}$.

In general, we want X to be large, so we are interested in a upper bound on the size of A-codes. We will see a theorem about this in section 4.

Example. For $n = 3$, we can consider the set X consisting of the three vectors $x_1 := (1, 0, 0)$, $x_2 := \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $x_3 := \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. We need now to compute the scalar products between them: $\langle x_1, x_2\rangle = 0$, $\langle x_1, x_3\rangle = \frac{1}{\sqrt{3}}$ and $\langle x_2, x_3\rangle = 0$, so the set X is a spherical $\left\{0, \frac{1}{\sqrt{3}}\right\}$-code.

The idea for the following definition of spherical designs is that sets that approximate the sphere globally well should approximate integrals over the sphere.

Definition. A subset $X \subseteq S^{n-1}$ is a spherical t-design if for any polynomial F with n variables and degree at most t we have

$$\frac{1}{\mathrm{vol}(S^{n-1})} \int_{S^{n-1}} F(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n = \frac{1}{|X|} \sum_{x \in X} F(x_1, \ldots, x_n),$$

where $\mathrm{vol}(S^{n-1}) = \int_{S^{n-1}} 1 \mathrm{d}x$.

Here, we would like our set X to be small, which is the reason why we are going to look at a lower bound on the size of t-designs in section 4.

Remark. In the following we will sometimes omit the word "spherical" and just talk about A-codes and t-designs.

Example. For $n = 2$ we claim that the set $\{(1,0),(-1,0)\}$ is a spherical 1-design.

Proof. Let $F(x_1, x_2) = ax_1 + bx_2 + c$ be an arbitrary polynomial of degree at most 1. Then we have

$$\frac{1}{\mathrm{vol}(S^1)} \int_{S^1} F(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 = \frac{1}{2\pi}\int_0^{2\pi} F(\sin(t), cos(t))\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}(a\sin(t) + b\cos(t) + c)\mathrm{d}t = \frac{1}{2\pi}2\pi c = c$$

and

$$\frac{1}{|X|}\sum_{x\in X} F(x_1, x_2) = \frac{1}{2}(F(1,0) + F(-1,0)) = \frac{1}{2}(a + c - a + c) = c.$$

Thus, $\frac{1}{\mathrm{vol}(S^1)}\int_{S^1} F(x)\mathrm{d}x = \frac{1}{|X|}\sum_{x\in X} F(x)$ for all polynomials F of degree at most 1. □

Example. Still for $n = 2$, we can in general say that a regular N-gon on S^1 is a t-design for $1 \le t \le N - 1$.

Example. For $n = 3$ we can give the following examples:

- 2-designs: Tetrahedron
- 3-designs: Cube and octahedron
- 5-designs: Dodecahedron and Icosahedron

Example. For $n = 4$, there is a 2-design $X \subseteq S^3$, which is also a $\{-\frac{1}{3}, \frac{1}{6}\}$-code, that corresponds to the Petersen graph, in the sense that $x_i \sim x_j$ if and only if $\langle x_i, x_j\rangle = -\frac{1}{3}$. This is an example of a more general construction: We can construct 2-designs that are $\{a, b\}$-codes from strongly regular graphs. However, this needs quite some setup, so we will not go into detail here.

Example. For $n = 24$, the set of minimal vector of the Leech lattice is an 11-design. We will see later in section 5 that in some sense this is the "best" 11-design in 24 dimensions.

3 Gegenbauer Polynomials

For fixed $n \ge 2$, we need the following family of polynomials in 1 variable. Note that we are using n on purpose here as this will later correspond to the dimension.

Definition. The Gegenbauer polynomial of degree k $Q_k^{(n)}$ is defined by the following recursion

$$\lambda_{k+1}Q_{k+1}^{(n)}(x) = xQ_k^{(n)}(x) - (1 - \lambda_{k-1})Q_{k-1}^{(n)}(x),$$

where $\lambda_k = \frac{k}{n+2k-2}$, $Q_0^{(n)}(x) = 1$ and $Q_1^{(n)} = nx$.

Thus, we for example have $Q_2^{(n)}(x) = \frac{n+2}{2}(nx^2 - 1)$.

Claim. Any polynomial $F(x)$ can be written as $F(x) = \sum_{k=0}^{\infty} f_k^{(n)} Q_k^{(n)}(x)$ for some coefficients $f_k^{(n)}$, which are also called Gegenbauer coefficients.

Proof idea. Note that the degree of $Q_k^{(n)}(x)$ is k. This can be shown by induction using the recursive definition. Then, we can determine for a polynomial of degree d the coefficient $f_d^{(n)}$ and then iteratively all other coefficients. Note that the coefficient $f_k^{(n)}$ for k larger than the degree of the polynomials will be zero. □

We will use this strategy now the find the Gegenbauer coefficients of the polynomial $F(x) = x^2 + 3x - 1$.

Example. As described above we first compute the coefficient $f_2^{(n)}$. This coefficient has to be $f_2^{(n)} = \frac{2}{n(n+2)}$ as then we have $f_2^{(n)}Q_2^{(n)}(x) = x^2 - \frac{1}{n}$ and so by subtracting it from $F(x)$ we can eliminate the quadratic term

$$F(x) - f_2^{(n)}Q_2^{(n)}(x) = 3x - 1 + \frac{1}{n}.$$

Using the same argument we get $f_1^{(n)} = \frac{3}{n}$ and then by subtracting again we get

$$F(x) - f_2^{(n)} Q_2^{(n)}(x) - f_1^{(n)} Q_1^{(n)}(x) = -1 + \frac{1}{n}.$$

Finally, $f_0^{(n)} = -1 + \frac{1}{n}$ and we can conclude that

$$F(x) = \frac{2}{n(n+2)} Q_2^{(n)}(x) + \frac{3}{n} Q_1^{(n)}(x) + \left(-1 + \frac{1}{n}\right) Q_0^{(n)}(x).$$

In the following we want to state two lemmas that we will need later to proof the bounds on the size of spherical codes and designs.

Definition. By $\mathrm{Hom}_n(k)$ we denote the polynomials in n variables that are homogeneous of degree k. By $\mathrm{Harm}_n(k) \subseteq \mathrm{Hom}_n(k)$ be denote the subspace of harmonic polynomials, i.e. those for which $\Delta F = 0$ holds.

Definition. For a finite set $\emptyset \neq X = \{\xi_1, \ldots, \xi_m\} \subseteq S^{n-1}$ and a orthonormal basis $\{W_j^k\}_{j=1}^{\dim(\mathrm{Harm}_n(k))}$ (where the scalar product is $\langle F, G \rangle = \frac{1}{\mathrm{vol}(S^{n-1})} \int_{S^{n-1}} F(x_1, \ldots, x_n) G(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n$) we define

- the k-th characteristic matrix H_k as $(H_k)_{i,j} = W_j^k(\xi_i)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, \dim(\mathrm{Harm}_n(k))$,
- for $\alpha \in [-1, 1]$ the distance matrix D_α as $(D_\alpha)_{i,j} = \begin{cases} 1 & \text{if } \langle \xi_i, \xi_j \rangle = \alpha \\ 0 & \text{otherwise.} \end{cases}$ for $i, j = 1, \ldots, m$,
- d_α as the sum of the elements of D_α.

We can now state the two lemmas we will need later. The proofs can be found in [DGS77, Corollary 3.8] and [DGS77, Theorem 5.3] respectively.

Lemma 1. *Let $X \subset S^{n-1}$ be a finite set with m elements. Also, let $A' = \{\langle \xi, \eta \rangle \mid \xi, \eta \in X\}$. Then for any polynomial $F(x)$ with Gegenbauer coefficients $f_0^{(n)}, f_1^{(n)}, \ldots$ it holds that*

$$f_0^{(n)} m^2 + \sum_{k=1}^{\infty} f_k^{(n)} ||H_k^T H_0||^2 = \sum_{\alpha \in A'} F(\alpha) d_\alpha.$$

Lemma 2. *A finite set $X \subseteq S^{n-1}$ is a spherical t-design if and only if its characteristic matrices satisfy $H_k^T H_0 = 0$ for $k = 1, 2, \ldots, t$.*

4 Bounds on the Size of Spherical Codes and Designs

4.1 Upper Bound for Spherical Codes

We need the following definition.

Definition. A polynomial $F(x)$ is compatible with the set A if for any $\alpha \in A$ we have that $F(\alpha) \leq 0$.

We can now state and prove an upper bound on the size of spherical codes.

Theorem 3. *Let $F(x)$ be a polynomial that is compatible with the set A, with Gegenbauer coefficients $f_0^{(n)} > 0$ and $f_k^{(n)} \geq 0$ for all k. Then the cardinality m of any A-code $X \subseteq S^{n-1}$ satisfies*

$$m \leq \frac{F(1)}{f_0^{(n)}}.$$

Proof. Using the assumptions and Lemma 1 we get

$$f_0^{(n)} m^2 \leq f_0^{(n)} m^2 + \sum_{k=1}^{\infty} f_k^{(n)} ||H_k^T H_0||^2 \qquad (f_k^{(n)} \geq 0)$$

$$
\begin{aligned}
&= \sum_{\alpha \in A'} F(\alpha) d_\alpha && \text{(Lemma 1)} \\
&\leq F(1) d_1 && \text{(by compatibility } F(\alpha) \leq 0 \text{ for } \alpha \neq 1) \\
&= F(1) \cdot m && (\langle \xi, \eta \rangle = 1 \Leftrightarrow \xi = \eta).
\end{aligned}
$$

Now, dividing by $m f_0^{(n)}$ concludes the proof. □

Example. For a set $A \subseteq [-1, \beta]$ for some $\beta = 0$, we can use $F(x) = x - \beta$. This polynomial is compatible with A and has Gegenbauer coefficients $f_0^{(n)} = -\beta > 0$ and $f_1^{(n)} = \frac{1}{n} \geq 0$. Thus by Theorem 3 the size of a spherical A-code is at most $\frac{1-\beta}{-\beta} = 1 - \frac{1}{\beta}$.

4.2 Lower Bound for Spherical Designs

We will now state and prove a lower bound on the size of spherical designs.

Theorem 4. *Let $F(x)$ be a polynomial, with Gegenbauer coefficients $f_0^{(n)} > 0$ and $f_k^{(n)} \leq 0$ for all $k > t$, that satisfies $F(1) > 0$ and $F(\alpha) \geq 0$ for all $\alpha \in [-1, 1]$. Then the cardinality m of any t-design $X \subseteq S^{n-1}$ satisfies*

$$
m \geq \frac{F(1)}{f_0^{(n)}}.
$$

Proof. Using Lemmas 1 and 2 as well as the assumptions we get

$$
\begin{aligned}
f_0^{(n)} m^2 &= f_0^{(n)} m^2 + \sum_{k=1}^{t} f_k^{(n)} ||H_k^T H_0||^2 && \text{(Lemma 2)} \\
&\geq f_0^{(n)} m^2 + \sum_{k=1}^{\infty} f_k^{(n)} ||H_k^T H_0||^2 && (0 \geq f_k^{(n)} \text{ for } k > t) \\
&= \sum_{\alpha \in A'} F(\alpha) d_\alpha && \text{(Lemma 1)} \\
&\geq F(1) d_1 && (F(\alpha) \geq 0 \text{ for } \alpha \in A' \subseteq [-1, 1]) \\
&= F(1) m && \text{(as above } d_1 = m).
\end{aligned}
$$

Again, by dividing by $m f_0^{(n)}$ we can conclude the theorem. □

Using Theorem 4 we can get the following two corollaries.

Corollary 5. *Let $X \subseteq S^{n-1}$ be a $(2e)$-design. Then $m = |X| \geq \binom{n+e-1}{n-1} + \binom{n+e-2}{n-1}$.*

Corollary 6. *Let $X \subseteq S^{n-1}$ be a $(2e+1)$-design. Then $m = |X| \geq 2\binom{n+e-1}{n-1}$.*

Proof idea. Apply Theorem 4 with $F(x) = \sum_{i=0}^{e} Q_i^{(n)}(x)$ respectively $F(x) = (x+1)\left(\sum_{i=0}^{\lfloor \frac{e}{2} \rfloor} Q_{e-2i}(n)(x)\right)^2$. □

5 Tight Designs and Their Classification

We will now move on to define tight spherical designs and state two theorems about their classification.

Definition. A spherical t-design is called tight if the bound from Corollary 5 respectively Corollary 6 is achieved.

Let us revisit two examples of t-designs we had before.

Example. For $n = 2$ we showed that $\{(1,0), (-1,0)\}$ is a 1-design. In this case it is a $(2e+1)$-design for $e = 0$. The bound from Corollary 6 is in this case $2\binom{2+0-1}{2-1} = 2$. Thus, $\{(1,0), (-1,0)\}$ is a tight 1-design.

Example. For $n = 3$ we said that the cube is a 3-design. Here, we need to use Corollary 6 with $e = 1$. In this case the bound is $2\binom{3+1-1}{3-1} = 6 < 8$. So, the cube is not a tight 3-design.

We now state a theorem about the classification of tight designs.

Theorem 7 (Bannai-Damerell)**.** *Assume $n \geq 3$. If a tight spherical t-design exists on S^{n-1}, then $t \in \{1, 2, 3, 4, 5, 7, 11\}$. Moreover, if $t = 11$, then $n = 24$ and $|X| = 196560$.*

The last case of the above theorem is in itself a theorem.

Theorem 8 (Bannai-Sloane)**.** *There is a (up to orthognal transformations) unique tight 11-design on $S^{23} \subseteq \mathbb{R}^{24}$, namely the 196560 minimal vectors of the Leech lattice.*

For $t = 1, 2, 3$ more is known on the classification of tight designs:

- Tight 1-designs on S^{n-1} are two antipodal points.
- Tight 2-designs on S^{n-1} are regular $(n + 1)$-simplices.
- Tight 3-designs on S^{n-1} are cross polytopes, i.e. sets of the form $\{\pm e_i \mid 1 \leq i \leq n\}$ for $e_1, \ldots, e_n$ an orthonormal basis of $\mathbb{R}^n$. An example of a cross polytope is the octahedron in 3 dimensions.

For $t = 4, 5, 7$ the classification of spherical t-designs is still an open problem.

References

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[BB09] Eiichi Bannai and Etsuko Bannai. "A survey on spherical designs and algebraic combinatorics on spheres". In: *European J. Combin.* 30.6 (2009), pp. 1392–1425. ISSN: 0195-6698. DOI: 10.1016/j.ejc.2008.11.007. URL: https://doi.org/10.1016/j.ejc.2008.11.007.