Seminar on Sphere Packings Lattices and Codes: Talk 11, Linear Programming Bounds for Sphere Packings II

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Today's talk is heavily focused on a paper by Henry Cohn and Noam Elkies from 2002 [1]. They developed a method to upper bound the maximum sphere packing density in any dimension relying on the Poisson Summation Formula. We will start with recalling some definitions and properties from the earlier talks and then start by reviewing the Fourier transform and the Poisson summation formula. Next we will state the bounding theorem and look at a simple example following chapter 3 in [1]. After that we will take a look at conditions for a sharp bound following chapter 5 of [1] and retrace some numerical results made in chapter 7 of [1]. In the end we will take a short look at why the known lattice packings in dimensions 8 and 24 are likely unique.

1 Preliminaries

Recall the dual of a lattice:

Definition 1.1 (Dual Lattice) *For a given (full rank) lattice $\Lambda \subset \mathbb{R}^n$ we define the dual lattice to be the set of linear forms on Λ with integral value. Since we are in $\mathbb{R}^n$ we have a scalar product $\langle \cdot, \cdot \rangle$ and can identify the dual lattice with a subset of $\mathbb{R}^n$:*

$$\Lambda^* := \{y \in \mathbb{R}^n \mid \forall x \in \Lambda \colon \langle x, y \rangle \in \mathbb{Z}\}$$

We also saw that we have something that we called the determinant of the lattice, which will correspond to the volume of a fundamental region of a lattice in this talk.

Definition 1.2 (Covolume)

$$|\Lambda| := \mathrm{vol}(\mathbb{R}^n / \Lambda)$$

Additionally the following correspondence between a lattice and its dual holds:

$$1 = |\Lambda||\Lambda^*|$$

2 Fourier Transform and Poisson summation

As you might remember from your previous studies we can define the Fourier series for periodic functions. We can generalize that to multiple dimensions and even lattices. So if $\Lambda \subset \mathbb{R}^n$ is a lattice and $\forall u \in \Lambda$, $\forall x \in \mathbb{R}^n \colon f(x+u) = f(x)$ we can develop it into a Fourier Series

Definition 2.1 (Fourier Series) *A function $f \colon \mathbb{R}^n \to \mathbb{R}$ that is periodic on a lattice Λ can be developed into a Fourier Series*

$$F(u) := \sum_{x \in \Lambda^*} \tilde{f}(x) e^{2\pi i \langle x,u \rangle}$$

where $\tilde{f}$ is defined as

$$\tilde{f}(t) := \frac{1}{|\Lambda|} \int_{\mathbb{R}^n/\Lambda} f(x) e^{-2\pi i \langle x,t \rangle} dx.$$

For sufficiently nice functions f the series F converges to the original function. Confer the lecture notes on this topic by Oded Regev [5].

Recall the Fourier transform as we have already seen last week:

Definition 2.2 (Fourier Transform) *The Fourier transform of an L1-function $f \colon \mathbb{R}^n \to \mathbb{R}$ is defined as*

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,t \rangle} dx.$$

Do not confuse the Fourier transform with the coefficients of the Fourier series. We will always denote the Fourier transform with a hat. Pay attention that Cohn & Elkies choose a different sign convention regarding the sign of the exponent of the exponential function. However we stick to the more common version with the minus sign in the exponent. With these definitions out of the way we can formulate and prove the Poisson Summation formula following chapter 2.3. in [3]:

Proposition 2.3 *Let $\Lambda \subset \mathbb{R}^n$ be a lattice and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions:*

1. *$\int_{\mathbb{R}^n} |f(x)| dx < \infty$ (i.e. f is L1-integrable on $\mathbb{R}^n$)*
2. *The series $\sum_{x \in \Lambda} |f(x+u)|$ converges uniformly for all u belonging to a compact subset of $\mathbb{R}^n$.*
3. *The Series $\sum_{y \in \Lambda^*} \widehat{f}(y)$ is absolutely convergent.*

Then the following holds:

$$\sum_{x \in \Lambda} f(x) = \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y)$$

Proof We assume first that $\Lambda = \mathbb{Z}^n$ and make a transformation argument later to extend this to general lattices. Let's look at the function

$$F(u) := \sum_{x \in \Lambda} f(x+u).$$

According to condition 2 this series converges uniformly and therefore F is continuous. It is also periodic on the lattice $\Lambda = \mathbb{Z}^n$: $\forall x \in \mathbb{Z}^n$: $F(u+x) = F(u)$.

Now we can develop its Fourier series, do not be confused by us seemingly summing over the lattice itself ($\mathbb{Z}^n$), in fact we are still summing over the dual lattice, but Z^n is self dual:

$$\sum_{y \in \mathbb{Z}^n} e^{2\pi i \langle u,y \rangle} \tilde{F}(y),$$

where $\tilde{F}(y) := \int_{[0,1]^n} F(t) e^{2\pi i \langle y,t \rangle} dt$. Now these $\tilde{F}$ are the Fourier series coefficients of the periodic function F, but we'll soon see that they are equal to the Fourier transform of the original function f, on the dual lattice points, i.e. $\forall y \in \Lambda^*$:

$$\tilde{F}(y) = \widehat{f}(y)$$

If we assume this equality for a little while we can see that because of Condition 3 we have that the Fourier series converges absolutely and uniformly, and thus to a continuous function, and hence to F. Now let's evaluate F at 0.

$$F(0) = \sum_{y \in \mathbb{Z}^n} e^{2\pi i \langle 0,y \rangle} \widehat{f}(y) = \sum_{y \in \mathbb{Z}^n} \widehat{f}(y)$$

But by definition of F we also have that

$$F(0) = \sum_{x \in \mathbb{Z}^n} f(x+0) = \sum_{x \in \mathbb{Z}^n} f(x),$$

and thus we get the Poisson summation formula

$$\sum_{x \in \mathbb{Z}^n} f(x) = \sum_{y \in \mathbb{Z}^n} \widehat{f}(y).$$

Now let's come back to why the Fourier series coefficients of F equal

the Fourier Transform of f on Λ^*: $\forall y \in \Lambda^*$:

$$\begin{aligned}
\tilde{F}(y) &= \int_{[0,1]^n} F(t) e^{-2\pi i \langle t,y \rangle} dt \\
&= \int_{[0,1]^n} \sum_{x \in \mathbb{Z}^n} f(x+t) e^{-2\pi i \langle t,y \rangle} dt \\
&= \int_{[0,1]^n} \sum_{x \in \mathbb{Z}^n} f(x+t) e^{-2\pi i \langle t+x,y \rangle} dt \\
&= \sum_{x \in \mathbb{Z}^n} \int_{x+[0,1]^n} f(t') e^{-2\pi i \langle t',y \rangle} dt' \\
&= \widehat{f}(y)
\end{aligned}$$

First we replace F by its definition and then we observe that $e^{-2\pi i m}$ will be equal to 1 whenever $m \in \mathbb{Z}$. Using the identification of the dual lattice using the scalar product as $\Lambda^* = \{y \in \mathbb{R}^n \mid \forall x \in \Lambda \colon \langle x, y \rangle \in \mathbb{Z}\}$, we can multiply by 1 and get $e^{-2\pi i(\langle t,y \rangle + \langle x,y \rangle)}$, as $x \in \Lambda$ and $y \in \Lambda^*$.

After exchanging integral and sum we can then shift the integral by the lattice points and get a tiling of the whole space $\mathbb{R}^n$ by the lattice fundamental cells. (Here: $[0,1]^n$)

This concludes the proof under our assumption that $\Lambda = \mathbb{Z}^n$. In the general case there is a matrix $M \in \mathrm{GL}_n(\mathbb{R})$ such that $\Lambda = M \cdot \mathbb{Z}^n$. Without loss of generality we can assume the matrix determinant to be positive, since we can simply rearrange the basis vectors. Using our dual lattice identification from earlier we can calculate that $\Lambda^* = \left(M^t\right)^{-1} \cdot \mathbb{Z}^n$ by observing that

$$\begin{aligned}
\Lambda^* &= \{\bar{y} \in \mathbb{R}^n \mid \forall x \in \mathbb{Z}^n (i.e. M \cdot x \in \Lambda) \colon \langle M \cdot x, \bar{y}, \in \rangle \mathbb{Z}\} \\
&= \{\bar{y} \in \mathbb{R}^n \mid \forall x \in \mathbb{Z}^n \colon \langle x, M^t \cdot \bar{y} \rangle \in \mathbb{Z}\}.
\end{aligned}$$

and noting that the last condition is equivalent to $\bar{y} = \left(M^t\right)^{-1}$ for some $y \in \mathbb{Z}^n$.

Writing down what we know so far:

$$\sum_{\bar{x} \in \Lambda} f(\bar{x}) = \sum_{x \in \mathbb{Z}^n} f(Mx) = \sum_{x \in \mathbb{Z}^n} f_M(x) = \sum_{y \in \mathbb{Z}^n} \widehat{f_M}(y)$$

Where we just defined the function f_M to be a variant of f where we first left multiply the argument by M.

Let's calculate the transform of f_M then for $y \in \mathbb{Z}^n$:

$$\begin{aligned}
\widehat{f_M}(y) &= \int_{\mathbb{R}^n} f(Mt) e^{-2\pi i \langle t, y \rangle} dt \\
&= \frac{1}{\det(M)} \int_{\mathbb{R}^n} f(t') e^{-2\pi i \langle (M^{-1}t'), y \rangle} dt' && \left(t = M^{-1}t'\right) \\
&= \frac{1}{|\Lambda|} \int_{\mathbb{R}^n} f(t') e^{-2\pi i \langle t', \left(M^{-1}\right)^t y \rangle} dt' \\
&= \frac{1}{|\Lambda|} \widehat{f}\left(\left(M^t\right)^{-1} y\right) && \left(\left(M^{-1}\right)^t = \left(M^t\right)^{-1}\right)
\end{aligned}$$

This means it is evaluating $\widehat{f}$ on the lattice points of the dual lattice, since $\Lambda^* = \left(M^t\right)^{-1} \cdot \mathbb{Z}^n$. Thus we can finaly conclude what we set out for:

$$\sum_{x \in \Lambda} f(x) = \sum_{y \in \mathbb{Z}^n} \widehat{f_M}(y) = \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y). \qquad \square$$

Why is this remarkable formula helpful? As an example we can use it to calculate a solution to the Basel problem

Proposition 2.4 (Basel problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The following proof was pointed out to me by Kaj Bäuerle and can be found in many other resources, such as an online blog of Zilin Jiang [4].

Proof We'd like to find a function f that gives us the stated series when summed over. A natural approach is to use $f(n) = \frac{1}{n^2}$ but we will see in a minute that that does not work out. So we think a bit harder and come up with

$$f(x) = \frac{1}{x^2 + c^2},$$

for some $c > 0$. After the calculation we will let $c \to 0$ and get the desired result.

The Fourier Transform can be calculated to be

$$\widehat{f}(y) = \frac{\pi}{c} e^{-2\pi c|y|}.$$

Let's check the three conditions so that we can do Poisson summation with the lattice $\Lambda = \mathbb{Z}$. Condition 1 demands $L1$-integrability. As we already succeeded in determining the Fourier Transform this condition is fulfilled. Condition 2 wants that $\sum_{x \in \mathbb{Z}} |f(x+u)|$ converges uniformly. This

also holds since the denominator is of high enough degree. Finally we need that the sum of the Fourier transform converges absolutely:

$$\sum_{y\in\mathbb{Z}} |\widehat{f}(y)| = \sum_{y\in\mathbb{Z}} |\frac{\pi}{c} e^{-2\pi c|y|}| < \infty$$

Looking carefully at the terms we notice that the exponential terms tend to 0 fast enough (faster than $O(n^{-1+\varepsilon})$) to fulfill the Condition 3.

Thus we can apply the summation formula and get

$$\begin{aligned}\sum_{x\in\mathbb{Z}} \frac{1}{x^2+c^2} &= \sum_{x\in\mathbb{Z}} f(x) \\ &= \sum_{y\in\mathbb{Z}} \widehat{f}(y) \\ &= \sum_{y\in\mathbb{Z}} \frac{\pi}{c} e^{-2\pi c|y|} \\ &= \frac{\pi}{c}\frac{e^{2\pi c}+1}{e^{2\pi c}-1}\end{aligned}$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+c^2} = \frac{1}{2}\left(\frac{\pi}{c}\frac{e^{2\pi c}+1}{e^{2\pi c}-1} - \frac{1}{c^2}\right)$$

Now we can take the limit of $c \to 0$, which produces the desired series on the left hand side and

$$\lim_{c\to 0^+} \frac{1}{2}\left(\frac{\pi}{c}\frac{e^{2\pi c}+1}{e^{2\pi c}-1} - \frac{1}{c^2}\right) = \frac{\pi^2}{6}$$

as claimed. □

For the purpose of this talk we'll define a slightly different condition for the validity of the Poisson Summation Formula:

Definition 2.5 *We call a function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *admissible* $\iff \exists c, \varepsilon > 0 \forall x \in \mathbb{R}^n$ *s.t.*

$$\begin{aligned}&|f(x)| < c\,(1+|x|)^{-n-\varepsilon}\text{, and} \\ &|\widehat{f}(x)| < c\,(1+|x|)^{-n-\varepsilon}.\end{aligned}$$

These bounds give us all the three conditions, we can calculate the Fourier transform and the sum $\sum_{y\in\Lambda^*} \widehat{f}(y)$ is absolutely convergent (conditions 1 and 2). But comparing with the bound on f we can also see that $\sum_{x\in\Lambda} f(x+u)$ is absolutely convergent regardless of the value of u and

thus uniformly convergent. For a more careful argument see Corollary 2.6 Chapter VII, p. 252 [6].

We will also only be interested in this equality which we can take directly from our proof of the Poisson summation formula: Let $f\colon R^n \to \mathbb{R}$ be an admissible function, $\Lambda \subset \mathbb{R}^n$ any lattice and $v \in \mathbb{R}^n$ any vector, then

$$\sum_{x\in\Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{y\in\Lambda^*} \widehat{f}(y) e^{2\pi i\langle v,y\rangle} \tag{2.1}$$

3 Cohn-Elkies bound for sphere packing density

Now that we have made ourselves familiar with the Poisson Summation Formula let's go back to the sphere packing problem and move on to the main theorem of the paper of Cohn & Elkies [1].

Let's familiarize ourselves with the center density first. We saw that for a lattice packing (i.e. each lattice point is the center of a sphere of radius $r/2$ where r is the length of a shortest nonzero vector) the packing density is given by

$$\Delta_\Lambda = \frac{1}{|\Lambda|} \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \left(\frac{r}{2}\right)^n,$$

where Γ denotes the Gamma function and

$$\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \left(\frac{r}{2}\right)^n = \mathrm{vol}\left(B_{r/2}(0)\right)$$

denotes the volume of a ball of radius $r/2$ in n dimensions.

One can think of this as the equivalent of one ball fitting into a fundamental cell of the lattice. Now we switch back to the setting of packings using unit balls (for a given lattice we can simply scale it so that a shortest vector has length 2) and define the so called 'center density' as the number of ball centers per *unit volume*.

Definition 3.1 (Center Density) *Given this discussion we define the center density δ for packings using unit balls using the following equation*

$$\Delta = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \delta$$

When using the formula for the density of a lattice packing and canceling out we get for a lattice packing of radius $r/2$ the following center density: $(r/2)^n / |\Lambda|$.

Next we need to be a bit careful how we handle our packings. First of all any general packing of course does not need to be a lattice packing, it even does not need to be periodic. But we can get arbitrarily close

to the greatest packing density in $\mathbb{R}^n$ for any given n using a periodic packing (c.f. Appendix A of [1]). Any periodic packing can be defined as a lattice packing with finitely many translates of that lattice. So any periodic packing is defined by a lattice $\Lambda \subset \mathbb{R}^n$ and a non empty set of vectors $v_1, \dots v_N \in \mathbb{R}^n$ such that none of their differences is a lattice point.

Theorem 3.2 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an admissible function that is not identical to 0. If f fulfills*

1. *$f(x) \leq 0$ for $|x| \geq 1$, and*
2. *$\widehat{f}(y) \geq 0$, for all y,*

then the center density of n-dimensional sphere packings is bounded from above by

$$\frac{f(0)}{2^n \widehat{f}(0)}.$$

If $\widehat{f}(0) = 0$ we interpret the equality as $+\infty$ giving no meaningful bound (but still satisfying the statement).

Proof As discussed it is enough to prove this for periodic packings, so let $\Lambda \subset \mathbb{R}^n$ be a lattice and $v_1, \dots, v_N \in \mathbb{R}^n, N > 0$ be a set of vectors such that none of their differences give a lattice point. By rescaling (which does not change the center density) we can limit ourselves to the case where a shortest lattice vector has length 1 and our packing thus contains balls of radius $1/2$. The center density is then given by

$$\delta = \frac{N}{2^n |\Lambda|}.$$

Now let's use the Poisson summation formula given by equation 2.1:

$$\forall v \in \mathbb{R}^n\colon \sum_{x \in \Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y) e^{2\pi i \langle v, y \rangle}$$

Let's play with the sums by summing twice over all translation vectors.

$$\begin{aligned} \sum_{1 \leq j,k \leq N} \sum_{x \in \Lambda} f(x + v_j - v_k) &= \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y) \sum_{1 \leq j,k \leq N} e^{2\pi i \langle v_j - v_k, y \rangle} \\ &= \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y) \left| \sum_{1 \leq j \leq N} e^{2\pi i \langle v_j, y \rangle} \right|^2 \end{aligned}$$

By condition 2 we know that $\widehat{f}$ is nonnegative so every term on the right hand side is bounded from below by the summand $y = 0$ which equals

$N^2\widehat{f}(0)/|\Lambda|$. On the other side of the equation $x + v_j - v_k$ is the difference between two centers of the packing which cannot get closer than 1, thus $|x + v_j - v_k| < 1 \Leftrightarrow x = 0 \wedge j = k$. Therefore condition 1 asserts that whenever $|x + v_j - v_k| \geq 1$ the term is nonpositive and we get an upper bound for the entire sum by $Nf(0)$. Thus,

$$Nf(0) \geq \frac{N^2\widehat{f}(0)}{|\Lambda|}$$

which we can rearrange recalling the definition of the center density (equation 3) to

$$\delta \leq \frac{f(0)}{2^n\widehat{f}(0)}.$$ □

We observe that the proof is entirely invariant under rotations, thus we can assume f to be a radial function.

Let's look at an example for $n = 1$. We define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := (1 + |x|)\,\chi_{[-1,1]}(x),$$

where $\chi_{[-1,1]}$ is the characteristic function on the unit disk. How did we get to this function? We observe that it is the convolution of $\chi_{[-1/2,1/2]}$ with itself, i.e.

$$\chi_{[-1,1]}(x) = \int_{\mathbb{R}} \chi_{[-1/2,1/2]}(t) \cdot \chi_{[-1/2,1/2]}(x-t)dt$$

The Fourier transform of $\chi_{[-1,1](x)}$ can be calculated using the convolution theorem to be

$$\left(\frac{\sin(\pi t)}{\pi t}\right)^2.$$

In order to use the theorem we need to check the conditions. Since $f(x) = 0$ whenever $|x| \geq 1$ it fulfills the first condition 1 and $\widehat{f}$ is also nonnegative for $t \neq 0$ and can be continuously extended at $t = 0$ to 1.

Admissibility follows by both functions being bounded by 1 with bounded support in the case of f and decaying sufficiently fast enough in the case of $\widehat{f}$. Therefore we get a bound for the center density by using $\Lambda = \mathbb{Z}$ and one translation vector $v_1 = 0$:

$$\delta \leq \frac{f(0)}{2^n\widehat{f}(0)} = \frac{1}{2}.$$

We notice that this is a sharp bound, since the packing defined by the lattice $2\mathbb{Z}$ attains the maximum packing density of 1.

In the following discussion we'll use a variant of this theorem:

Theorem 3.3 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an admissible function fulfilling three conditions:*

1. $f(0) = \widehat{f}(0) > 0$,
2. $f(x) \leq 0$ *for* $|x| \geq r$, *and*
3. $\widehat{f}(y) \geq 0$ *for all* t.

Then the center density of unit sphere packings in $\mathbb{R}^n$ is bounded from above by $(r/2)^n$.

This is a scaled variant of the first theorem and can be proved either by being careful in regard to scaling in proof of the first theorem or directly as shown in Th 3.2. [1].

Before we concern ourselves with the question how to find such suitable functions f we first make some observations about the conditions for a sharp bound.

4 Conditions for a sharp bound

As mentioned in the beginning this chapter follows closely chapter 5 in [1].

Most of the well-known examples of the best lattice packings in dimensions 2, 8 and 24 (the A_2 root lattice in dimension 2, E_8 root lattice in dimension 8 and the Leech lattice in dimension 24) are equal to their dual (so called *isodual* lattices). So they must have covolume 1. This is not true in dimension 2, but we can rescale A_2 so that it is *isodual* using a rotation.

Suppose Λ is an isodual lattice of covolume $|\Lambda| = 1$ and f is a radial function fulfilling the conditions in theorem 3.3 giving a sharp bound, i.e. r is the length of a shortest nonzero vector in Λ. Then applying Poisson summation yields

$$\sum_{x\in\Lambda} f(x) = \sum_{y\in\Lambda^*} \widehat{f}(y).$$

After substracting $f(0) = \widehat{f}(0)$ from both sides we are left with only nonpositive terms on the left hand side and nonnegative terms on the right hand side, demanding that $\forall x \in \Lambda \setminus \{0\}$ and $\forall y \in \Lambda^* \setminus \{0\}$ $f(x) = \widehat{f}(y) = 0$.

One can naturally try to construct the function f by a the knowledge of the zeros attained above. In one dimension the following f works:

$$f(x) = (1-x^2)\prod_{k\geq 2}\left(1-\frac{x^2}{k^2}\right)^2 = \frac{1}{1-x^2}\left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

This has the right zeros and also fulfills $f(x) \leq 0$ for $|x| \geq 1$. If we were to go ahead and compute the Fourier transform we would even get that $\widehat{f} \geq 0$.

It has support $[-1,1]$ and is positive in $(-1,1)$. It gives a tight bound, so it is another way to prove the optimality of the known sphere packings in 1 dimension. However it is hard to generalize this approach. One could replace the sine function by a Bessel function, but that does not give sharp bounds for $n > 1$.

One thing to note here is that $\widehat{f}$ cannot have compact support (i.e. bounded support) for $n > 1$. If it did, then the zeros of f could not be sufficiently densely-spaced. More precisely: If $\widehat{f}$ is a ranotdial function with support inside of the ball $B_R(0)$ of radius R around the origin then the common value of $f(r)$ for vectors of length r satisfies

$$f(r) = \int_{B_R(0)} \widehat{f}(t) e^{2\pi i \langle rx,t \rangle} dt,$$

where $x \in \mathbb{R}^n$ is any vector with $|x| = 1$. Thus we get an entire function of r and for all $r \in \mathbb{C}$:

$$|f(r)| \leq e^{2\pi R|r|} \int_{B_R(0)} \left| \widehat{f}(t) \right| dt$$

so f is a function of exponential type. One can use Jensen's formula to argue that f can have at most linearly spaced zeros, but the nonzero vectors in the Leech lattice have lengths $\sqrt{2k}$ for integers $k > 1$ and those in E_8 have lengths $\sqrt{2k}$ for integers $k > 0$. Since f needs to vanish at these vector lengths, the roots are too densely spaced for $\widehat{f}$ to have compact support.

Maybe the whole restriction to radial functions is too limiting? This seems unlikely since we can simply rotate a lattice and f must still vanish on the points in order to give a sharp bound.

5 Numerical results and conjectures in dimension 2, 8 and 24

Again as mentioned earlier this section follows closely chapter 7 in [1].

Regardless of our fruitless attempts in the previous chapter we can now do some numerical optimization to improve on the best known bounds (in 2002). We keep in mind that the length of a shortest vector in a given lattice gives us the density of the corresponding lattice packing. Thus if we can show that in a given lattice $\Lambda \subset \mathbb{R}^n$ there is a vector of length r we have bounded the density of the best sphere packing in that dimension.

Therefore we can examine the following variant of our theorem for isodual lattices.

Proposition 5.1 *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a radial, admissible function, not identical to zero, satisfying the following three properties:*

1. $g(0) = 0$,
2. $g(x) \geq 0$, *for* $|x| \geq r$ *for some* $r > 0$,
3. $\widehat{g} = -g$.

Then every isodual lattice in dimension n must contain a nonzero vector of length at most r.

Note that condition 3 is well defined since we can directly map each lattice point to a linear form in the dual lattice (isometrically). With this proposition the problem becomes finding a suitable radial function g and minimizing the distance r from the origin at which it stays nonnegative for increasing r.

Proof (of special case) For simplicity, as it suffices for our numerical calculations, we assume $g(x) > 0$ if $|x| \gg 0$. Let $\Lambda \cong \Lambda^*$ be an isodual lattice. Then we use Poisson summation, notice that $|\Lambda| = 1$:

$$\sum_{x \in \Lambda} g(x) = \sum_{y \in \Lambda^*} = -\sum_{x \in \Lambda} g(x),$$

so

$$\sum_{x \in \Lambda} g(x) = 0.$$

But we know from condition 2 that for all $|x| > r$ the terms must be nonnegative. Thus together with our additional assumption (that there are positive terms for larger r) in order for the sum to vanish the lattice must contain a vector of length at most r where g is allowed to attain a negative value. □

About the relationship with the upper bound theorems: For any given function fulfilling the conditions of theorem 3.3 we can set $g = \widehat{f} - f$, thus the bound for isodual lattices is at least as good as for general sphere packings.

Now we want to construct a suitable function g. Conceptually we take certain polynomials that we multiply with a Gaussian to decay fast enough for $r \to \infty$. Let $L_k^\alpha(x)$ be the so called Laguerre polynomials orthogonal with respect to the measure $e^{-x}x^\alpha dx$ on $[0, \infty)$. Set $\alpha = n/2 - 1$ and define for $k \geq 0$:

$$g_k(x) := L_k^\alpha\left(2\pi|x|^2\right) e^{-\pi|x|^2}$$

These functions form a basis for the radial eigenfunctions of the Fourier transform, with eigenvalues $(-1)^k$. Now consider a linear combination of $g_1, g_3, \ldots g_{4m+3}$ and require it to have a root at 0 and double roots at $z_1, \ldots z_m$. Now we can calculate the $r > 0$ for which the function g makes the last change of sign. Together with an initial guess for the z_i we have

an optimization problem where we can locally minimize the value of r via small perturbations of the initial z_i. When increasing m this method should give better results.

With this method we can achieve results for isodual lattices, but there's no reason that the best sphere packing should be given by an isodual lattice. But we can transform our calculated functions g and make them applicable for the setting of theorem 3.3 without changing r as follows: Let h be a linear combination of the remaining $g_0, g_2, \ldots, g_{4m+2}$ with double zeros at $z_1, \ldots, z_m$ such that $g + h$ has a double zero at 0. In the examples that Cohn & Elkies computed $g + h$ then has constant sign, without loss of generality positive. Therefore $f := -(g + h)$ is nonpositive outside radius r with Fourier transform $\widehat{f} = g + h$ that is nonnegative everywhere and $f(0) = \widehat{f}(0)$. Now with theorem 3.3 we get the same bound for general sphere packings that g proves for isodual lattices. They also note that it is not clear that this f fulfills the rest of the conditions for theorem 3.3 but it turns out that f does that in their concrete calculations.

Their results for $m = 11$ in dimensions 11 and 24 come very close to the optimal density of the known packings. In dimension 8 with

$$2\pi r^2 = 12.56637375,$$

they are within a factor of 1.000001 of equality and in dimension 24 with

$$2\pi r^2 = 25.1342216$$

they are within a factor of 1.0007071 of equality. See the tables in chapter 7 of [1] for a listing of the final values for the roots $z_1, \ldots, z_m$ as well as numerical results for r in other dimensions.

This leads to the following conjectures:

Conjecture 5.2 (Conjecture 7.2. in [1]) *The smallest possible value r in proposition 5.1 equals that in Theorem 3.3, and for each optimal g from proposition 5.1, there exists an optimal f from Theorem 3.3 such that* $g = \widehat{f} - f$.

Conjecture 5.3 (Conjecture 7.3. in [1]) *There exists funtions that satisfy the hypothesis of Theorem 3.3 and solve the sphere packing problem in dimensions 2, 8 and 24.*

This conjecture will be the main topic in our concluding talk next week, where we'll see a construction in the dimensions 8 and 24. However according to Danylo Radchenko the problem is still open in dimension 2.

Conjecture 5.4 (Conjecture 7.4. in [1]) *The numerical method described above gives bounds that converge (as* $m \to \infty$*) to the optimal bounds obtainable using Theorem 3.3.*

6 conditions for uniqueness of optimal packing

This section follows closely chapter 8 of [1].

It is conjectured that the E_8 and Leech lattices give a unique (among periodic sphere packings) packing achieving maximum density. We will discuss an argument for this based on a slightly stronger version of Conjecture 5.3:

Let Λ_2, Λ_8 and Λ_{24} denote the isodual scaling of the hexagonal lattice, the E_8 root lattice and the Leech lattice respectively.

Conjecture 6.1 *For $n \in \{2, 8, 24\}$, there exists a function that satisfies the hypothesis of Theorem 3.3 to prove that Λ_n is the densest packing in $\mathbb{R}^n$. Furthermore, this function and its Fourier transform have roots only at the vector lengths in Λ_n.*

Let n be 8 or 24, so that Λ_n is a unimodular lattice and f a function fulfilling this conjecture 6.1. Suppose we have a maximally dense packing using a lattice Λ and translates $v_1, \ldots, v_N$, whose differences are not in Λ. Without loss of generality we can scale the lattice so that $|\Lambda| = N$ and assume $v_1 = 0$.

Now we can again apply Poisson summation and get

$$\sum_{1 \le j,k \le N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{y \in \Lambda^*} \widehat{f}(y) \left| \sum_{1 \le j \le N} e^{2\pi i \langle v_j, y \rangle} \right|^2 .$$

Again the terms $f(0) = \widehat{f}(0)$ cancel each other out (remember we are in the setting of Theorem 3.3), and we get in particular that each vector $x + v_j - v_k$ must be a root for f. Now we use the following lemma:

Lemma 6.2 *Let $S \subset \mathbb{R}^n$ such that $0 \in S$ and there are n linearly independent vectors in S. If $\forall x, y \in S \; \exists k \in 2\mathbb{Z}^{\ge 0}$ s.t. $|x - y| = \sqrt{k}$ then the subgroup of $\mathbb{R}^n$ generated by S is an even integral lattice.*

Recall that in an even integral lattice the scalar product of two lattice vectors is an integer and the norm of each lattice vector is even.

Proof For all $x, y \in S$ we have $\langle x, y \rangle \in \mathbb{Z}$ since

$$\langle x, y \rangle = \left(|x - 0|^2 + |y - 0|^2 - |x - y|^2 \right) / 2 \in \mathbb{Z},$$

and the norm $|x|^2$ of any element in S is an even integer (using that $0 \in S$). The same holds then for integer linear combinations of the elements of S.□

Now we can use the strengthening property that f has roots only at the vector lengths in Λ_n. That is at $r = \sqrt{2k}$ for $k > 0$ in dimension 8 and for $k > 1$ in dimension 24 and since f needs to vanish on all points in Λ we

have that the length of the vectors in Λ needs to be square roots of even integers as well.

Thus the subgroup $L \subset \mathbb{R}^n$ created by our periodic packing in question is an even integral lattice following the Lemma.

Now we have that in an integral lattice the covolume must always be the square root of an integer, so $|\Lambda| \in \left\{1, \sqrt{2}, \sqrt{3}, 2, \dots\right\}$. And thus L can have at most one point per unit volume with exactly one point if the lattice is unimodular, i.e. $|\Lambda| = 1$. However since we know the density of the packing and $|\Lambda| = N$ the period packing has one sphere per unit volume. Thus $N = 1$ and the periodic packing is given by the (unimodular) lattice Λ. Now it needs to have shortest vectors equal to those in Λ_8 and Λ_{24} respectively in order to attain the maximum density.

Such lattices are unique (see Chapters 16 and 18 in [2]).

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