Overview and motivation

Danylo Radchenko

March 3, 2021
The sphere packing problem

Problem (The sphere packing problem)

What is the best way to pack equal spheres in $\mathbb{R}^d$?
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More precisely, we are interested in covering as large a portion of the space as possible by non-overlapping spheres (or balls) of radius $R$. 
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More precisely, we are interested in covering as large a portion of the space as possible by non-overlapping spheres (or balls) of radius $R$.

A precise formulation of this problem requires some care (for example in defining what exactly is the portion of the space covered by spheres).
The first known formulation of the sphere packing problem (in 3 dimensions) goes back to Johannes Kepler’s manuscript “The Six-Cornered Snowflake” from 1611. His interest in arrangements of spheres came from his correspondence with an English astronomer Thomas Harriot. Harriot himself was studying the problem of how to best stack cannonballs at the behest of Sir Walter Raleigh.
The first known formulation of the sphere packing problem (in 3 dimensions) goes back to Johannes Kepler’s manuscript “The Six-Cornered Snowflake” from 1611. His interest in arrangements of spheres came from his correspondence with an English astronomer Thomas Harriot. Harriot himself was studying the problem of how to best stack cannonballs at the behest of Sir Walter Raleigh.
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A formal proof verification using automated proof checking software was completed in 2014.
Optimal sphere packing in $\mathbb{R}^2$
Optimal sphere packing in $\mathbb{R}^3$
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Optimal sphere packings in $\mathbb{R}^3$

- hexagonal close-packed
- face-centered cubic
Known results for the sphere packing problem

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The Leech lattice is of a different nature and it was first constructed from a certain exceptional error-correcting code.
The problem of the thirteen spheres

In 1694 Isaac Newton and David Gregory have discussed the following question.

**Question**

*Can a sphere touch 13 spheres of the same size?*
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One can easily arrange 12 spheres to touch a given sphere, but there is enough room to move them around.
A configuration of 12 spheres touching another one
The kissing number problem

In general, one can ask the same question in $d$ dimensions.

Problem (The kissing number problem)

What is the maximal number of unit spheres that can touch a given unit sphere in $\mathbb{R}^d$?
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This is a spherical analogue of the sphere packing problem.
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Problem (The kissing number problem)

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This is a spherical analogue of the sphere packing problem.

A slightly different formulation is that we are interested in finding the maximal value of $N$ for which there exists points $x_1, \ldots, x_N \in S^{d-1}$ satisfying

$$\langle x_i, x_j \rangle \leq 1/2 \quad \text{for all} \ i \neq j$$

Any such set $\{x_1, \ldots, x_N\}$ is called a kissing configuration and the maximal value of $N$ (for given $d$) is called the kissing number.
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With the exception of $d = 4$ this table mirrors that for the sphere packing problem. For $d = 8$ and $d = 24$ the best kissing configuration is unique and comes from the exceptional lattices ($E_8$ and the Leech lattice).
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A generalization of the question about kissing configurations is the question about spherical codes.
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**Definition (Spherical codes)**

Given a subset $S \subseteq [-1, 1)$, we call a set of points $\{x_1, \ldots, x_N\} \subseteq \mathbb{S}^{d-1}$ a spherical $S$-code, if one has

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Similarly to the kissing number problem one can ask what is the maximal size of a spherical $S$-code.
An interesting case arises already when $|S| = k$ is finite. Then spherical codes are $k$-distance sets, i.e. there are only $k$ distinct distances that occur between $x_i$'s.
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Delsarte, Goethals and Seidel have found a bound on the size of spherical $S$-codes that depends only on $k = |S|$. They also found that the same bound serves as a lower bound on the size of spherical designs.

**Definition (Spherical designs)**

A spherical $t$-design is a set $\{x_1, \ldots, x_N\} \subseteq S^{d-1}$ such that

$$\frac{1}{N} \sum_{i=1}^{N} p(x_i) = \int_{S^{d-1}} p(x) d\mu(x)$$

for all polynomials $p$ of degree $\leq t$. 

As it turns out, a spherical $S$-code that achieves the Delsarte-Goethals-Seidel bound must also be a strong spherical design. Such configurations that achieve these bounds are called **tight spherical designs**.
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It turns out that the shortest vectors of the $E_8$ lattice and the Leech lattice are particularly nice tight designs. This fact is also related to their optimality as kissing configurations.
There is a discrete side to the story, which surprisingly has to do with the mathematical theory of communication as laid out by Claude Shannon in 1948.
Transmission of information

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A very simple way in which one can control the errors over a noisy channel is to group the bits into blocks, and add to each block an additional bit equal to the parity of the sum in the block.
Error-correcting codes

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In general, we represent bits by elements of the field with two elements $\mathbb{F}_2$. Then blocks of $n$ bits are elements of the vector space $\mathbb{F}_2^n$. One introduces the **Hamming distance** between two elements $x, y \in \mathbb{F}_2^n$ as

$$d(x, y) = \#\{1 \leq i \leq n: x_i \neq y_i\}$$
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An $[n, k, d]$-binary linear code is a linear subspace $V \subseteq \mathbb{F}_2^n$ of dimension $k$ such that any two distinct vectors in $V$ have Hamming distance $\geq d$. 

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One can easily recognize that error-correcting codes are analogues of sphere packings for the Hamming distance. Moreover, one can construct (good) sphere packings from (good) codes.
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There are several ways of doing this, the easiest is to embed the codewords as elements of the set $\{0, 1\}^n \subset \mathbb{R}^n$ and extend to a $(2\mathbb{Z})^n$-periodic set.
There is a particularly interesting binary linear code that was discovered by Marcel J. E. Golay in 1949. This is the so-called extended binary Golay code with parameters $[24, 12, 8]$. It has several remarkable features: It is the unique binary code with parameters $[24, 12, 8]$. Its automorphism group is a sporadic simple group (the Mathieu group $M_{24}$). If one removes the last coordinate one obtains a perfect $[23, 12, 7]$ binary code (Hamming balls of radius 3 cover $F_2^{23}$).
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**Definition (Steiner system)**

A Steiner system $S(n, k, l)$ is a set $S$ of $k$-subsets of $X = \{1, \ldots, n\}$ such that any $l$-subset of $X$ is contained in exactly one element of $S$. 
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As a (rather exceptional) example: a Steiner system $S(24, 8, 5)$ is given by the codewords of the extended binary Golay code with eight 1's.
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Curiously, the construction of this lattice was not given much special prominence, and its description occupied less than a page in a supplement to his 1964 paper.
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Sporadic simple groups are a finite list of 26 (or 27) groups that occupy an exceptional place in the classification of finite simple groups (among cyclic groups, alternating groups, and 16 families of groups of Lie type).
The final topic we will discuss in the seminar is modular forms. Modular forms are certain special analytic functions on $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Modularity forms are a very deep subject with many interesting number-theoretic connections. A reason for this is a combination of two facts: many “interesting” arithmetic sequences are encoded by modular forms, and spaces of modular forms are finite-dimensional and “computable.”
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$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

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- many “interesting” arithmetic sequences are encoded by modular forms
- spaces of modular forms are finite-dimensional and “computable”
To give an example, let us consider the sum of $k$ squares function

$$r_k(n) = \#\{(a_1, \ldots, a_k) \in \mathbb{Z}^k : a_1^2 + \cdots + a_k^2 = n\}$$
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Then the generating series
\[ \theta^k(\tau) = \sum_{n \geq 0} r_k(n)q^n \]
becomes a modular form of weight $k/2$ if we set $q = e^{2\pi i \tau}$ (though not for the full group $\text{SL}_2(\mathbb{Z})$). This is a simplest example of a theta function.
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As an application: knowing a basis for the corresponding space of modular forms for $k = 4$ allows one to deduce the following identity (due to Jacobi)

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This implies Lagrange’s four-square theorem (\( r_4(n) > 0 \)).
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These topics will be the subject of the last four talks of the seminar.