Overview and motivation

Danylo Radchenko

March 3, 2021

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More precisely, we are interested in covering as large a portion of the space as possible by non-overlapping spheres (or balls) of radius R.

A precise formulation of this problem requires some care (for example in defining what exactly is the portion of the space covered by spheres).

The Kepler Conjecture

The first known formulation of the sphere packing problem (in 3 dimensions) goes back to Johannes Kepler's manuscript "The Six-Cornered Snowflake" from 1611. His interest in arrangements of spheres came from his correspondence with an English astronomer Thomas Harriot. Harriot himself was studying the problem of how to best stack cannonballs at the behest of Sir Walter Raleigh.

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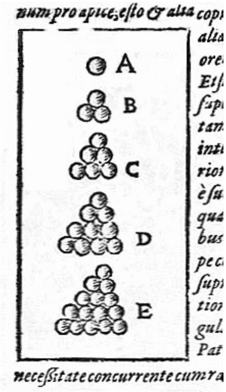

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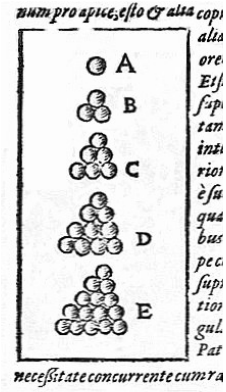

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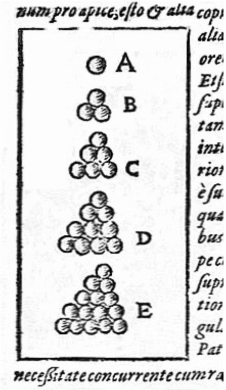

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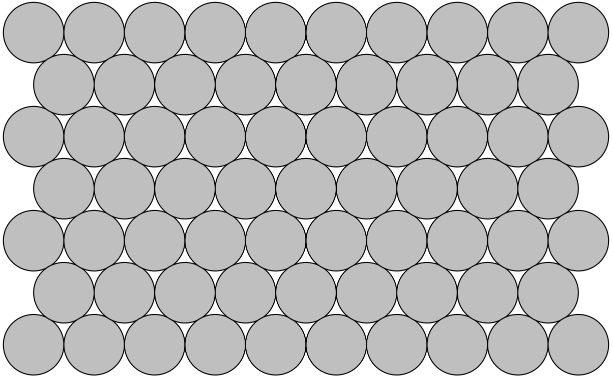
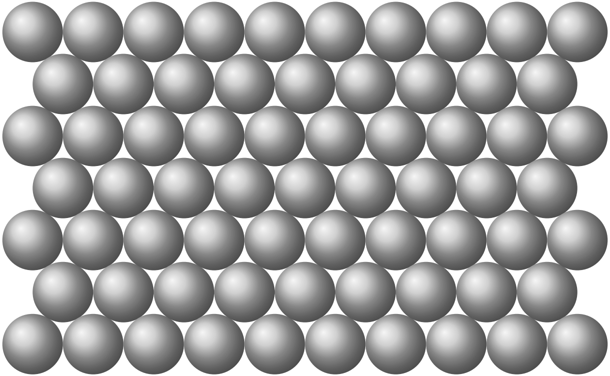
A formal proof verification using automated
proof checking software was completed in 2014.

Optimal sphere packing in $\mathbb{R}^2$



Optimal sphere packing in $\mathbb{R}^3$



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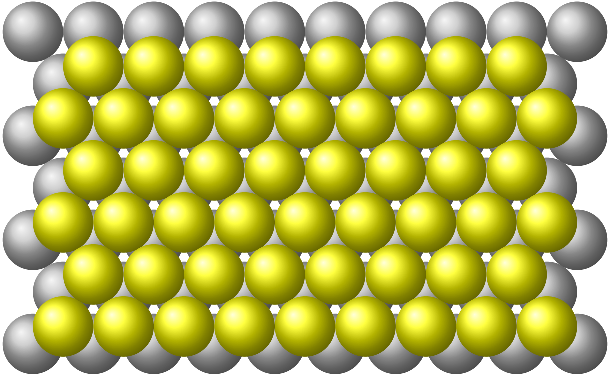

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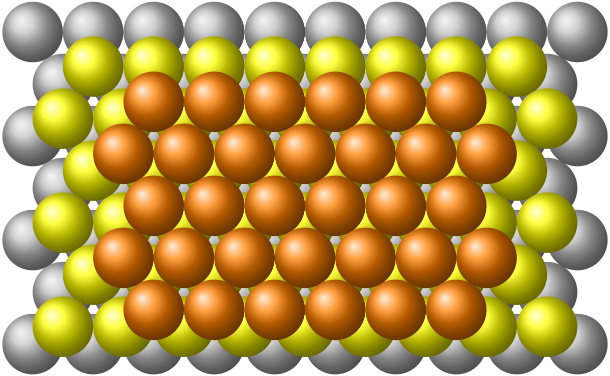
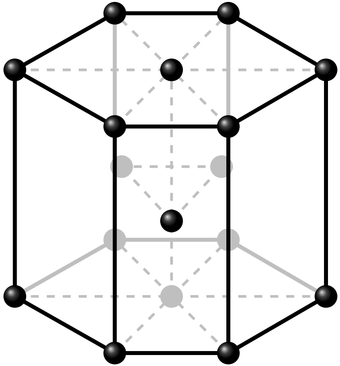
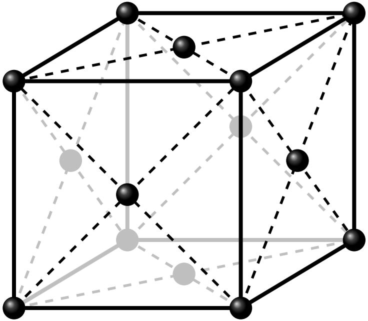


Optimal sphere packings in $\mathbb{R}^3$



hexagonal close-packed



face-centered cubic

Known results for the sphere packing problem

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For $d \leq 8$ the best packing should be achieved by a certain root lattice.
These arise in the classification of finite-dimensional simple Lie algebras.

The Leech lattice is of a different nature and it was first constructed from a certain exceptional error-correcting code.

The problem of the thirteen spheres

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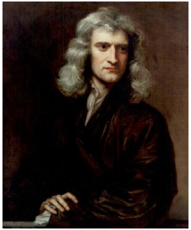

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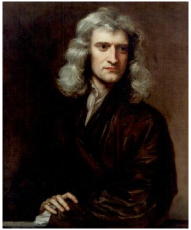
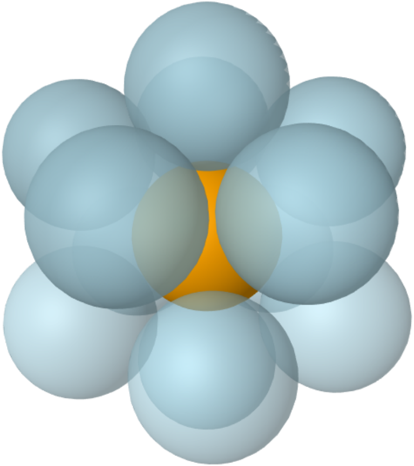


One can easily arrange 12 spheres to touch a given sphere, but there is enough room to move them around.

A configuration of 12 spheres touching another one



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This is a spherical analogue of the sphere packing problem.

A slightly different formulation is that we are interested in finding the maximal value of N for which there exists points $x_1, \ldots, x_N \in \mathbb{S}^{d-1}$ satisfying

$$\langle x_i, x_j \rangle \leq 1/2 \quad \text{for all } i \neq j$$

Any such set $\{x_1, \ldots, x_N\}$ is called a **kissing configuration**
and the maximal value of N (for given d) is called the **kissing number**.

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For $d = 8$ and $d = 24$ the best kissing configuration is unique and comes from the exceptional lattices (E_8 and the Leech lattice).

Spherical codes and spherical designs

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Definition (Spherical codes)

Given a subset $S \subseteq [-1,1)$, we call a set of points $\{x_1, \ldots, x_N\} \subseteq \mathbb{S}^{d-1}$ a spherical S-code, if one has

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Similarly to the kissing number problem one can ask what is the maximal size of a spherical S-code.

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An interesting case arises already when $|S| = k$ is finite. Then spherical codes are k-distance sets, i.e. there are only k distinct distances that occur between x_i's.

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Delsarte, Goethals and Seidel have found a bound on the size of spherical S-codes that depends only on $k = |S|$. They also found that the same bound serves as a lower bound on the size of spherical designs.

Definition (Spherical designs)

A spherical t-design is a set $\{x_1, \ldots, x_N\} \subseteq \mathbb{S}^{d-1}$ such that

$$\frac{1}{N}\sum_{i=1}^{N} p(x_i) = \int_{\mathbb{S}^{d-1}} p(x)d\mu(x)$$

for all polynomials p of degree $\leq t$.

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It turns out that the shortest vectors of the E_8 lattice and the Leech lattice are particularly nice tight designs. This fact is also related to their optimality as kissing configurations.

Transmission of information

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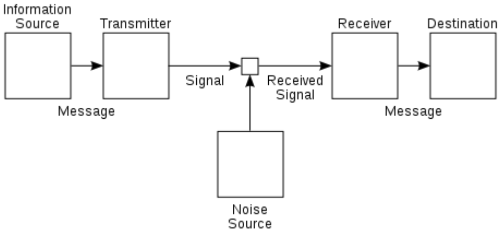

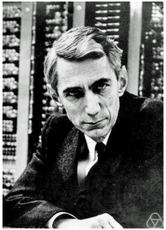

Error-correcting codes

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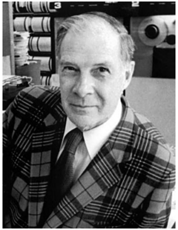
It turns out, that one can add more control bits to also be able to detect where the error has occurred and automatically fix it. That this is possible was realized by Richard Hamming, and he also constructed the first error-correcting code: the Hamming (7,4) code.



Error-correcting codes

In general, we represent bits by elements of the field with two elements $\mathbb{F}_2$. Then blocks of n bits are elements of the vector space $\mathbb{F}_2^n$. One introduces the **Hamming distance** between two elements $x, y \in \mathbb{F}_2^n$ as

$$d(x, y) = \#\{1 \leq i \leq n \colon\ x_i \neq y_i\}$$

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Hamming's (7,4)-code is a binary linear $[7,4,3]$-code.

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One can easily recognize that error-correcting codes are analogues of sphere packings for the Hamming distance. Moreover, one can construct (good) sphere packings from (good) codes.

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There are several ways of doing this, the easiest is to embed the codewords as elements of the set $\{0,1\}^n \subset \mathbb{R}^n$ and extend to a $(2\mathbb{Z})^n$-periodic set.

Binary Golay code

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It has several remarkable features:

- It is the unique binary code with parameters $[24, 12, 8]$
- Its automorphism group is a sporadic simple group (the Mathieu group M_{24})
- If one removes the last coordinate one obtains a **perfect** $[23, 12, 7]$ binary code (Hamming balls of radius 3 cover $\mathbb{F}_2^{23}$)

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As a (rather exceptional) example: a Steiner system $S(24, 8, 5)$ is given by the codewords of the extended binary Golay code with eight 1's.

The Leech lattice

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Curiously, the construction of this lattice was not given much special prominence, and its description occupied less than a page in a supplement to his 1964 paper.

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The group of symmetries of the Leech lattice was analyzed by J. H. Conway in 1968, and as a result he found three new sporadic simple groups that now bear his name.

Sporadic simple groups are a finite list of 26 (or 27) groups that occupy an exceptional place in the classification of finite simple groups (among cyclic groups, alternating groups, and 16 families of groups of Lie type).

Modular forms

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$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

plus some growth conditions. Here $k \in \mathbb{Z}$ is called the **weight** of the modular form f.

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plus some growth conditions. Here $k \in \mathbb{Z}$ is called the **weight** of the modular form f.

Modular forms are a very deep subject with many interesting number-theoretic connections. A reason for this is a combination of two facts

- many "interesting" arithmetic sequences are encoded by modular forms
- spaces of modular forms are finite-dimensional and "computable"

Modular forms

To give an example, let us consider the sum of k squares function

$$r_k(n) = \#\{(a_1, \dots a_k) \in \mathbb{Z}^k \colon a_1^2 + \cdots + a_k^2 = n\}$$

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Then the generating series

$$\theta^k(\tau) = \sum_{n \geq 0} r_k(n) q^n$$

becomes a modular form of weight $k/2$ if we set $q = e^{2\pi i \tau}$ (though not for the full group $\mathrm{SL}_2(\mathbb{Z})$). This is a simplest example of a theta function.

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As an application: knowing a basis for the corresponding space of modular forms for $k = 4$ allows one to deduce the following identity (due to Jacobi)

$$r_4(n) = 8 \sum_{4 \nmid d \mid n,} d\,, \quad n \geq 1\,.$$

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This implies Lagrange's four-square theorem ($r_4(n) > 0$).

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These topics will be the subject of the last four talks of the seminar.