Sphere packings, Lattices and Codes

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1. **The sphere packing problem.**

   Statement of the problem. Definition and basic properties of lattices: fundamental region, discriminant, Gram matrix. Density of a lattice packing and of a general packing. Summary of results in small dimensions. Nonconstructive lower bounds for optimal density (density $\gg 2^{-n}$ for greedy packing and the Minkowski-Hlawka theorem [19, Ch. 3]).

   References: [6, Ch 1.1], [4, §1], [19, Ch. 3]

2. **Error-correcting codes.**

   Motivation: the error-correcting code problem. Definition of a code. Hamming distance between codewords, minimal distance, and rate of a code. Linear codes, cyclic codes. Examples: Binary Hamming codes, Reed-Solomon codes, Reed-Muller codes. Sphere packings and lattices from codes: Construction A and Construction B ([6, Ch. 5.2, Ch. 5.3] or [19, §5.2, §5.3])

   References: [6, Ch. 3.2], [19, Ch. 5]

3. **Shannon capacity of graphs.**

   Motivation: zero-error capacity of a channel. Definition of the Shannon capacity of a graph. Computation of capacity for small graphs. Lovasz's theorem on the Shannon capacity of the pentagon.

   Optional: Shannon capacity of the disjoint union of graphs ([12, Ch. 29]).

   References: [15], [11], [12, Ch. 28]

4. **Steiner systems and finite projective planes.**

   Definition of combinatorial block designs. Steiner systems. Divisibility conditions for existence of Steiner systems. Keevash's theorem [10] (only the statement about existence of Steiner systems, without proof). Finite projective planes and $S(2, n+1, n^2+n+1)$. Construction of finite projective planes of order p^k. Bruck-Ryser-Chowla theorem [9, Thm. 2.3] (with proof if there is enough time); nonexistence of finite projective planes of certain orders.

   References: [2, §1], [9, Ch. 1], [6, Ch. 3.3]

5. **The binary Golay code and the Leech lattice.**

   More on linear codes: weight enumerators ([8, §2.7]), quadratic residue codes. Constructions of the $[23, 12, 7]$-code: as a quadratic residue code ([19, Ex. 5.2]) and as a cyclic code ([14, p. 64]). Construction of the extended binary Golay code. Steiner system $S(5, 8, 24)$. Uniqueness of the $[24, 12, 8]$-code ([8, Thm. 2.6], with proof if there is enough time). Construction of the Leech lattice from the extended binary Golay code ([19, Ex. 5.9]); shortest vectors in the Leech lattice.

   References: [19, Ch. 5], [8, §2.7, §2.8]

6. **Spherical designs and spherical codes.**

   Motivation: quadrature formulas on spheres. Definition of spherical t-designs [7, §5]. Examples of spherical designs [1, §2.3]. Spherical codes. Upper bounds for spherical codes and lower bounds for spherical designs ([7, Thm. 4.3, Thm. 5.10]). Tight designs ([1, Def. 2.13]) and their classification ([1, Thm.2.16, Thm.2.17], without proofs). Equivalence between 2-designs with two distances and strongly regular graphs ([7, Ex. 9.1]).

References: [1], [7]

7. **The kissing number problem.**

   The Gregory-Newton problem ([19, §1.3]). General formulation of the kissing number problem and a summary of results in small dimensions. Kissing number problem in dimensions 8 and 24 ([19, §9.2]). Uniqueness of the kissing configuration in dimension 8 ([19, Thm. 9.3]). Musin's theorem (only outline of the proof given in [13, §2]).

   References: [19, Ch. 1, Ch. 9], [6, Ch. 1.2], [13]

8. **Linear programming bounds for sphere packings I.**

   Linear programming bounds for spherical codes (following [19, §8.2] or [7, Thm 4.8]). Kabatyanski-Levenshtein bound for spherical codes ([19, §8.3], [6, §9.3.5]). An improved geometric lemma relating sphere packing density and the maximal size of spherical codes ([5, Prop. 2.1]). Upper bounds for sphere packing density in $\mathbb{R}^n$.

   References: [19, Ch. 8], [6, Ch. 9], [5], [7].

9. **Modular forms for the group $\mathrm{PSL}_2(\mathbb{Z})$.**

   The modular group and the fundamental domain. Definition of modular forms. Valence formula and the dimension formula for $M_k(\mathrm{PSL}_2(\mathbb{Z}))$. Eisenstein series and the structure theorem for $M_*(\mathrm{PSL}_2(\mathbb{Z}))$. The modular discriminant function $\Delta(\tau)$.

   References: [18, §1], [16, Ch. VII.1-4]

10. **Theta functions of lattices.**

    Further properties of lattices: integrality, unimodularity. Even and odd lattices. Dual lattice and the Poisson summation formula. Theta function of $\mathbb{Z}^n$; application: Jacobi's identity for the sum-of-four-squares function. Theta function of an even unimodular lattice. Applications of modularity: dimension of an even integral lattice is divisible by 8; number of vectors of length $\sqrt{2n}$ in the Leech lattice (and congruence mod 691 for $\tau(n)$); the notion of extremal even unimodular lattices.

    Optional: E_8 is the unique even unimodular lattice in $\mathbb{R}^8$ (using the classification of root systems, see [6, Ch. 4.2]);

    References: [18, §3], [16, Ch. VII.6]

11. **Linear programming bounds for sphere packings II.**

    Fourier transform and the Poisson summation formula. Cohn-Elkies bound for the sphere packing density ([3, §3]). Conditions for a sharp bound ([3, §5]). Description of numerical results and conjectures in dimensions 2, 8, and 24. Conditions for uniqueness of the optimal sphere packing ([3, §8]).

    References: [3]

12. **The sphere packing problem in 8 dimensions.**

    Modular forms for $\Gamma(2)$ and the Eisenstein series of weight 2 [17, Section 3]. Construction of Fourier eigenfunctions [17, Section 4]. Proof of sphere packing optimality of the E_8 lattice.

    References: [17]

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