CHAPTER 3

STANDARD POISSON PROCESS.

Motivations :

- theoretical importance :
  - → "simple" continuous - time stochastic process.
  - → "universal" stationary process. $\mathbb{R}_+ \to \mathbb{N}$ indep. increments and jump of size 1.
- applications
  - → queuing processes.
  - → compound poisson process.

Framework : . $(\Omega, \mathcal{F}, \mathbb{P})$ probability space

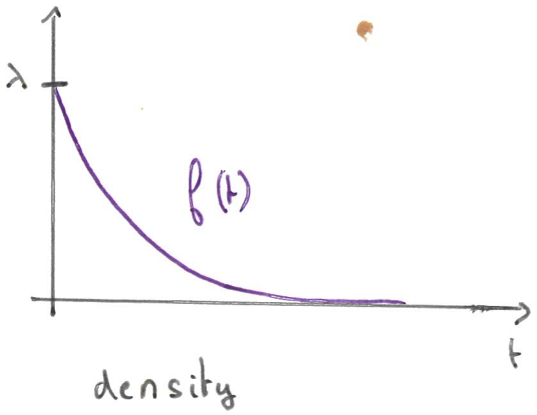
. time space $\mathbb{R}_+ = [0, \infty)$.

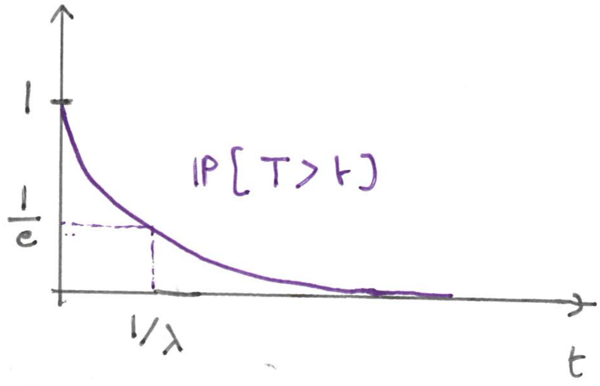
1 EXPONENTIAL RANDOM VARIABLES

Def. Let $\lambda > 0$. A real r.v. T is exponential with parameter λ (we write $T \sim \mathrm{Exp}(\lambda)$) if it has density

$$f(t) = \lambda e^{-\lambda t} \mathbb{1}_{t \geq 0} .$$

Rk: $T \sim \mathrm{Exp}(\lambda) \iff \forall t \geq 0 \quad P[T > t] = e^{-\lambda t}$.



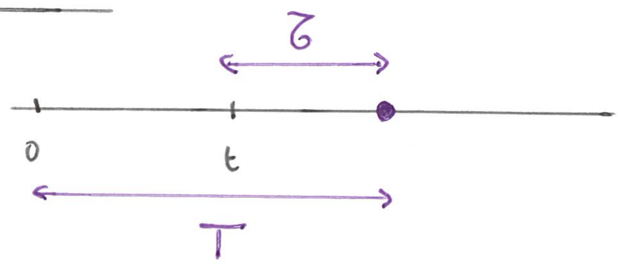


Prop. (memoryless property)

Let $T \sim \mathrm{Exp}(\lambda)$.

$$\boxed{\forall s, t \geq 0 \quad \mathbb{P}[T \geq s+t \mid T \geq t] = \mathbb{P}[T \geq s]}$$

Interpretation.



$\mathbb{P}[\tau \geq s \mid T > t] = e^{-\lambda s}$

"condition on $T \geq t$, $\tau := T - t$ is $\mathrm{Exp}(\lambda)$."

Prop. (min of indep. exponentials.)

Let $n \geq 0$, $T_1, \dots, T_n$ indep. $T_i \sim \mathrm{Exp}(\lambda_i)$ $\lambda_i > 0$.

- $\min(T_1, \dots, T_n) \sim \mathrm{Exp}(\lambda_1 + \dots + \lambda_n)$
- $\mathbb{P}[T_1 = \min(T_1, \dots, T_n)] = \dfrac{\lambda_1}{\lambda_1 + \dots + \lambda_n}$

<u>Reminder:</u> If X is a real r.v. with density f and Y is a r.v. with values in some measurable space $(E, \mathcal{E})$ independent of X. Then for all $\varphi : \mathbb{R} \times E \to \mathbb{R}$ meas. bounded, we have.

$$\boxed{\mathbb{E}[\varphi(X, Y)] = \int_{\mathbb{R}} \mathbb{E}[\varphi(x, Y)] f(x)\, dx}$$

heuristic: $\mathbb{E}[\varphi(X,Y)] = \int \underbrace{\mathbb{E}[\varphi(X,Y) \mid X = x]}_{\overset{?}{=}\mathbb{E}[\varphi(x,Y)] \text{ indep.}} \underbrace{P[X=x]\, dx}_{f(x)\, dx}$

<u>Proof:</u> • For every $t \geq 0$

$$\mathbb{P}[\min(T_1, \dots, T_n) \geq t] \overset{\text{indep.}}{=} \prod_{i=1}^{n} \mathbb{P}[T_i \geq t]$$

$$= \exp(-(\lambda_1 + \dots + \lambda_n) t)$$

• $\mathbb{P}[T_1 = \min(T_1, \dots, T_n)]$

$$\overset{\text{reminder}}{=} \int_0^\infty \mathbb{P}[t = \min(t, T_2, \dots, T_n)] \lambda_1 e^{-\lambda_1 t}\, dt$$

$$= \int_0^\infty \mathbb{P}[\min(T_2, \dots, T_n) \geq t]\, \lambda_1 e^{-\lambda_1 t}\, dt$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \dots + \lambda_n) t}\, dt = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_n} \qquad \blacksquare$$

Prop. (Sum of exponentials)

Let $\lambda > 0$, $n \geq 1$. Let $T_1, \dots, T_n$ iid $Exp(\lambda)$.

Then $S_n := T_1 + \dots + T_n$ is $\Gamma(n, \lambda)$-distributed.

ie. S_n is a continuous r.v. with density

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Rk: $\Gamma(1, \lambda) = Exp(\lambda)$.

Proof: By induction.

- $n=1$ ok.
- Assume that the result holds for some $n \geq 1$.

Let $T_1, \dots, T_{n+1}$ iid $Exp(\lambda)$. $S_n = T_1 + \dots + T_n$ and T_{n+1} are indep., hence $S_{n+1} = S_n + T_{n+1}$ admits a density given by the convolution

$$f_{S_{n+1}}(t) = \int_0^t f_{S_n}(s) f_{T_{n+1}}(t-s)\, ds$$

$$\overset{(IH)}{=} \int_0^t \lambda^2 e^{-\lambda s} \cdot \frac{(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda(t-s)} ds$$

$$= \dots = \lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

■

2. DEFINITION OF POISSON PROCESSES

Setup: • $\lambda > 0$.

• $(T_i)_{i \geq 1}$ iid $Exp(\lambda)$

• $S_n := T_1 + \dots + T_n$

Def. The stochastic process $(N_t)_{t \geq 0}$ defined by

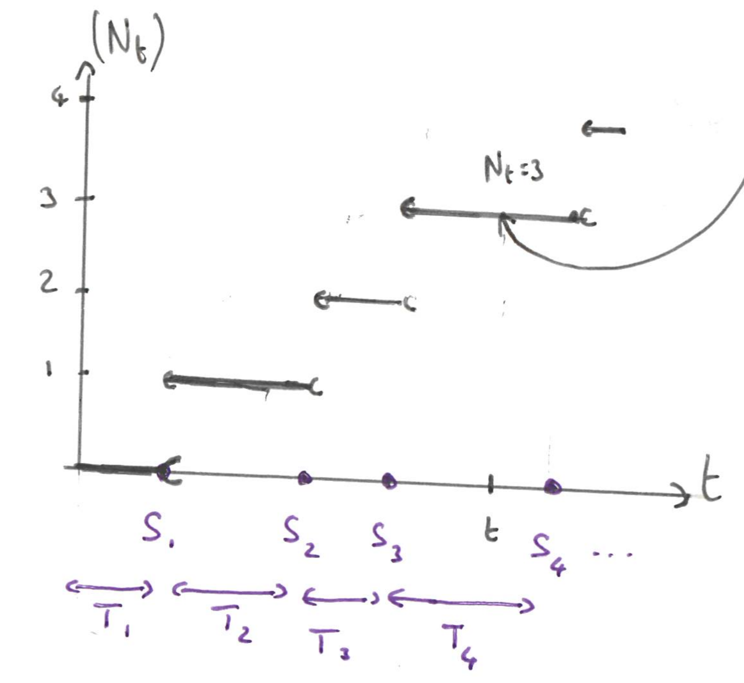
$$\boxed{\forall t \geq 0 \quad N_t = \sum_{i=1}^{\infty} \mathbb{1}_{S_i \leq t}}$$

is called Poisson process with intensity λ ($PP(\lambda)$)

The r.v. $T_1, T_2, \dots$ are the inter-arrival times

and $S_1, S_2, \dots$ are the jump times of the process.

Interpretation



N_t = "number of clock rings before time t"

Elementary properties

- the mapping $t \to N_t$ is a.s. right-continuous nondecreasing, with values in $\mathbb{N}$.

- For fixed $t \geq 0$ $\boxed{N_t \sim \text{Poiss}(\lambda t)}$

  i.e. $\mathbb{P}[N_t = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad (\forall n \in \mathbb{N})$

Proof: The first item follows from the definition. For the second item, first notice $\mathbb{P}[N_t = 0] = \mathbb{P}[T_1 > t] = e^{-\lambda t}$ and then

$\forall n \geq 1 \quad \mathbb{P}[N_t = n] = \mathbb{P}[S_n \leq t, \; S_n + T_{n+1} > t]$

S_n, T_{n+1} indep.

$$= \int_0^\infty \mathbb{P}[s \leq t, \; s + T_{n+1} > t] \, \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds$$

$$= \int_0^t \lambda e^{-\lambda(t-s)} e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad \blacksquare$$

Rk: other def. of Poisson processes exist in the litterature. The definition above ensures that Poisson processes exist. (because a sequence of iid exists)

The memoryless property of the exponentials implies a Markov property for the Poisson process.

<u>Notation</u>: Fix $t > 0$. Consider the process $N^{(t)} = (N_s^{(t)})_{s \geq 0}$ defined by $\forall s \geq 0$ $\boxed{N_s^{(t)} = N_{t+s} - N_t}$

<u>Thm</u> [Markov property for the Poisson processes]

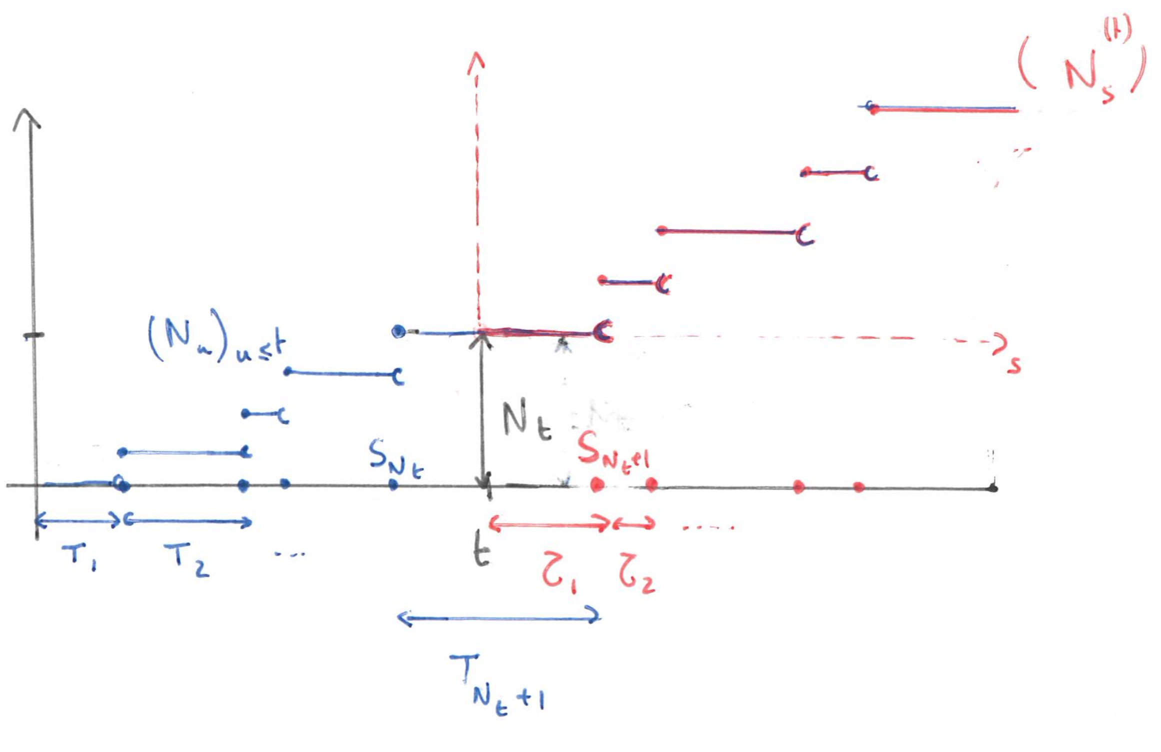
Fix $t > 0$. The process $N^{(t)}$ is a $PP(\lambda)$ independent of $(N_u)_{0 \leq u \leq t}$.

ie: There exist some $(\tau_i)_{i \geq 1}$ iid $Exp(\lambda)$ s.t.

$$N_s^{(t)} = \sum_{i=1}^{\infty} \mathbb{1}_{\tau_1 + \dots + \tau_i \leq t}$$

and for every $u_1, \dots, u_k \leq t$ $s_1, \dots, s_\ell \geq 0$

$$(N_{u_1}, \dots N_{u_k}) \text{ and } (N_{s_1}^{(t)}, \dots, N_{s_\ell}^{(t)}) \text{ indep.}$$



Proof: Define

$$\tau_1 = S_{N_t+1} - t \quad \text{and for } i \geq 2 \quad \tau_i = T_{N_t+i}$$

Step 1: $\tau_1 \sim \mathrm{Exp}(\lambda)$ independent of N_t

$$\forall s \geq 0 \ n \geq 1 \ \mathbb{P}[N_t = n, \tau_1 > s] = \mathbb{P}[S_n \leq t, S_{n+1} > t, S_{n+1} > t+s]$$

$$= \mathbb{P}[S_n \leq t, S_n + T_{n+1} > t+s]$$

$$= \int_0^t \underbrace{\mathbb{P}[u + T_{n+1} > t+s]}_{} f_{S_n}(u)\, du$$

$$= \mathbb{P}[T_{n+1} > (t-u)+s]$$

memoryless property of T_{n+1}

$$\overset{\downarrow}{=} e^{-\lambda s} \times \mathbb{P}[T_{n+1} > t-u]$$

$$= e^{-\lambda s} \int_0^t \mathbb{P}[u + T_{n+1} > t] f_{S_n(u)} du$$

$$= e^{-\lambda s}\, \mathbb{P}[S_n \leq t, S_{n+1} > t]$$

$$= e^{-\lambda s}\, \mathbb{P}[N_t = n]$$

(the case $n=0$ can be deduced from above, or proved directly by the memoryless property of T_1)

Step 2: $(\tau_i)_{i\geq 1}$ are iid $Exp(\lambda)$, indep. of N_t

$\forall i \geq 1 \ \forall s_1, \dots, s_i \geq 0 \quad \forall n \in \mathbb{N}$

$$\mathbb{P}[N_t = n, \tau_1 > s_1, \dots, \tau_i > s_i]$$

$$= \mathbb{P}[\underbrace{N_t = n, \tau_1 > s_1}_{\in \sigma(T_1, \dots, T_{n+1})}, \underbrace{T_{n+2} > s_2, \dots, T_{n+i} > s_i}_{\in \sigma(T_{n+2}, \dots, T_{n+i})}]$$

$$\overset{\text{indep.}}{=} \mathbb{P}[N_t = n, \tau_1 > s_1] \cdot e^{-\lambda(s_2 + \dots + s_i)}$$

$$\overset{\text{step 1}}{=} \mathbb{P}[N_t = n] \, e^{-\lambda(s_1 + \dots + s_i)}$$

Since $N_s^{(t)} = \sum_{i=1}^{\infty} \mathbb{1}_{\tau_1 + \dots + \tau_i \leq s}$ (for every $s \geq 0$), this concludes that $N^{(t)}$ is a $pp(\lambda)$ indep. of N_t.

Step 3: $N^{(t)}$ is independent of $(N_u)_{u \leq t}$.

In order to achieve Step 3, one can first use an induction (on k) and Step 2 to show that for every $k \geq 1$.

$\forall p > k \quad \forall \ 0 = t_0 < t_1 < \dots < t_p \quad \forall n_1, \dots, n_p \in \mathbb{N}$

$$\mathbb{P}[N_{t_1} = n_1, \dots, N_{t_p} = n_p] = \mathbb{P}[N_{t_1} = n_1, \dots, N_{t_k} = n_k] \times \mathbb{P}[N_{t_{k+1} - t_k} = n_{k+1} - n_k, \dots, N_{t_p - t_k} = n_p - n_k]$$

(exercise)

Now let $u_1 < \dots < u_k < t$ $\quad 0 \le s_1 < \dots < s_\ell$

$n_1, \dots, n_k \in \mathbb{N}$ $\quad m_1, \dots, m_\ell \in \mathbb{N}$

$\forall n \in \mathbb{N}$ the formula above implies

$$\mathbb{P}[N_{u_1} = n_1, \dots, N_{u_\ell} = n_\ell, N_t = n, N^{(t)}_{s_1} = m_1, \dots, N^{(t)}_{s_\ell} = m_\ell]$$

$$= \mathbb{P}[N_{u_1} = n_1, \dots, N_{u_\ell} = n_\ell, N_t = n]\, \underbrace{\mathbb{P}[N_{s_1} = m_1, \dots, N_{s_\ell} = m_\ell]}_{= \mathbb{P}[N^{(t)}_{s_1}, \dots, N^{(t)}_{s_\ell} = m_\ell]}$$

By summing over all $n \in \mathbb{N}$, we obtain the desired independence property.

4 INDEPENDENT AND STATIONARY INCREMENTS.

<u>Def.</u> A stochastic process $(X_t)_{t \ge 0}$ is said to have independent and stationary increments if

$\forall k \ge 1$ $\forall\, 0 = t_0 < \dots < t_k$ $\quad X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}}$ are independent

and

$\forall s < t$ $\forall h \ge 0$ $\quad X_t - X_s \overset{\text{law}}{=} X_{t+h} - X_{s+h}$

Thm (stationary and independent increments)

(i) $\forall k \quad \forall\, 0 = t_0 < t_1 < \dots < t_k$

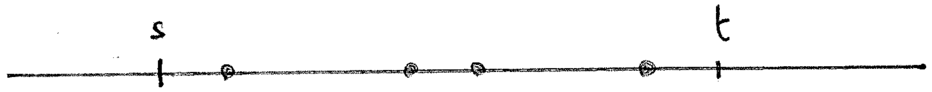
$$N_{t_1} - N_{t_0}, \; \dots, \; N_{t_k} - N_{t_{k-1}} \text{ are independent}$$

(ii) $\forall\, t, s \geq 0$

$$N_{t+s} - N_t \sim \text{Poiss.}(\lambda s)$$

In particular $(N_t)_{t \geq 0}$ has stationary and independent increments

Interpretation:



$$N_t - N_s = \text{number of arrivals in the interval } [s, t]$$

(i) $\Leftrightarrow$ the arrivals in disjoint intervals are independent.

(ii) $\Leftrightarrow$ the law of the number of arrivals depends only on the length of the interval

Rk: (i) and (ii) is equivalent to $\forall k \geq 1$.

$$\forall\, 0 = t_0 < \dots < t_k \qquad \forall\, n_1, \dots, n_k \in \mathbb{N}$$

(*) $$\mathbb{P}\left[N_{t_1} - N_{t_0} = n_1, \dots, N_{t_k} - N_{t_{k-1}} = n_k\right] = \prod_{i=1}^{k} \left(\frac{(\lambda(t_i - t_{i-1}))^{n_i}}{n_i!} e^{-\lambda(t_i - t_{i-1})} \right)$$

Proof: We prove (*) by induction on k.

- The case $k=1$ corresponds to $N_t \sim \text{Poiss.}(\lambda t)$.

- Now let $k \geq 1$ and assume that (*) holds.

Let $t_0 < \dots < t_{k+1}$ $n_1, \dots, n_{k+1} \in \mathbb{N}$

$$\mathbb{P}\Big[\underbrace{N_{t_1} - N_{t_0} = n_1, \dots,}_{\in \sigma((N_u)_{u \leq t_k})} \underbrace{N_{t_{k+1}} - N_{t_k}}_{= N^{(t_k)}_{t_{k+1}-t_k}} = n_{k+1}\Big]$$

$$\overset{MP}{=} \underbrace{\mathbb{P}\big[N_{t_1} - N_{t_0} = n_1, \dots, N_{t_k} - N_{t_{k-1}} = n_k\big]}_{\overset{\text{induction}}{=} \prod_{i \leq k} \frac{[\lambda(t_i - t_{i-1})]^{n_i}}{n_i!} e^{-\lambda(t_i - t_{i-1})}} \underbrace{\mathbb{P}\big[N_{t_{k+1}-t_k} = n_{k+1}\big]}_{= \frac{(\lambda(t_{k+1}-t_k))^{n_{k+1}}}{n_{k+1}!} e^{-\lambda(t_{k+1}-t_k)}}$$

$$= \prod_{i \leq k+1} \frac{[\lambda(t_i - t_{i-1})]^{n_i}}{n_i!} e^{-\lambda(t_i - t_{i-1})}$$

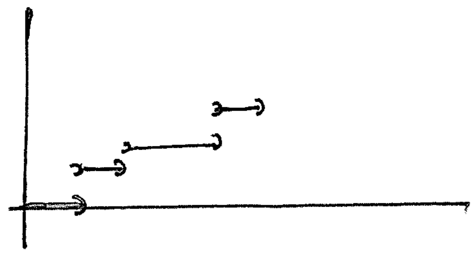
■

5 FINITE MARGINALS CHARACTERIZATION

Motivation : If N is a pp (λ), we know the law of the vector $(N_{t_1}, \dots, N_{t_k})$ for every fixed $t_1 < \dots < t_k$.
(called the finite marginal laws : they characterize the law of the stochastic process $(N_t)_{t\geq 0}$)

Conversely, if a stochastic process $(N_t)_{t \geq 0}$ has the same finite marginals as a p.p.(λ), is it a pp(λ) ?

$\hookrightarrow$ No : • For $T_1, T_2, \dots$ iid $\mathrm{Exp}(\lambda)$, consider the process $\tilde{N}_t = \sum_{i\geq 1} \mathbb{1}_{S_i < t}$ $\quad$ ($S_i = T_1 + \dots + T_i$)



$(\tilde{N}_t)_{t\geq 0}$ is not a pp. because it has not right-continuous trajectories a.s. But it has the same finite marginals as the p.p. defined by $N_t = \sum_{i \geq 1} \mathbb{1}_{S_i \leq t}$.

• One can even construct a "worse" counter-example, (inspired from W. Werner lecture notes on Brownian motion).
Let N be a p.p.(λ). Let $(X_i)_{i\in\mathbb{N}}$ iid $\mathrm{Exp}(1)$, indep. of N. Notice that $\mathcal{X} := \{X_1, X_2, \dots\}$ is dense in $\mathbb{R}_+$.

Consider the process defined by $\forall t$ $\tilde{N}_t = N_t + \mathbb{1}_{t \in \mathcal{X}}$

The process $\tilde{N}_t$ is not a pp(λ) (it is nowhere right continuous a.s.) but it has the same finite marginals.

↳ Yes : In this Section, we will see that if we add a regularity condition on $t \to N_t$. Then it is characterized by its finite marginal laws.

Def: Let $N := (N_t)_{t \geq 0}$ be a continuous-time stochastic process. We say that N is a counting process if the following holds a.s. :

$N_0 = 0$ and $t \mapsto N_t$ is non decreasing, right-continuous, with values in $\mathbb{N}$.

In this case we can define the successive jump times by setting $S_1 = \min\{t : N_t > 0\}$ and by induction for $i \geq 1$ $S_{i+1} = \min\{t \geq S_i : N_t > N_{S_i}\}$.

Rk: N is a pp(λ) if and only if

N is a counting process with jumps of size 1 [i.e. ($\forall t$ $\limsup\limits_{s \to 0} N_t - N_{t-s} \leq 1$) a.s] and

$S_1, S_2 - S_1, S_3 - S_2$ are iid $Exp(\lambda)$.

Thm: Let $\lambda > 0$. Let $N := (N_t)_{t \geq 0}$ be a counting process.

The following are equivalent.

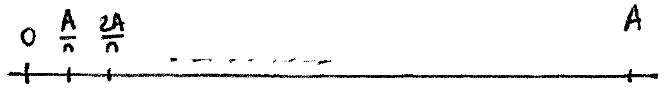
(i) N is a $PP(\lambda)$

(ii) $\forall k \geq 1 \quad \forall t_0 = 0 < t_1 < \dots < t_k \quad \forall n_1, \dots, n_k \in \mathbb{N}$

$$\mathbb{P}\left[N_{t_1} - N_{t_0} = n_0, \dots, N_{t_k} - N_{t_{k-1}} = n_k \right] = \prod_{i=1}^{k} e^{-\lambda(t_i - t_{i-1})} \frac{[\lambda(t_i - t_{i-1})]^{n_i}}{n_i!}$$

Proof: (i) $\Rightarrow$ (ii) already seen.

(ii) $\Rightarrow$ (i) We first prove that $(N_t)_{t \geq 0}$ has jumps of size 1 on every segment $[0, A]$, $A > 0$.



Let $E_n = \{ \forall i \leq n, \ N_{\frac{iA}{n}} - N_{\frac{iA}{n} - \frac{A}{n}} \leq 1 \}$ for $n \geq 1$.

We have $\mathbb{P}[E_n] = \prod_{i \leq n} \left(e^{-\frac{\lambda A}{n}} + e^{-\frac{\lambda A}{n}} \times \frac{\lambda A}{n} \right) = e^{-A} \left(1 + \frac{\lambda A}{n}\right)^n \underset{n \to \infty}{\longrightarrow} 1.$

Let $E = \bigcup_{n \geq 1} E_n$. We have $\mathbb{P}[E] = 1$ (because $\mathbb{P}[E] \geq \mathbb{P}[E_n]$ for all $n \geq 1$) and furthermore $\forall \omega \in E$

$\forall t \leq A \quad \limsup_{s \to 0} N_t(\omega) - N_{t-s}(\omega) \leq 1.$

This concludes that N has jumps of size 1.

Fix $k \geq 1$. We prove that $T_1 = S_1, T_2 = S_2 - S_1, \dots, T_k = S_k - S_{k-1}$ are iid $\mathrm{Exp}(\lambda)$

We begin with the computation of the law of $(S_1, \dots, S_k)$.

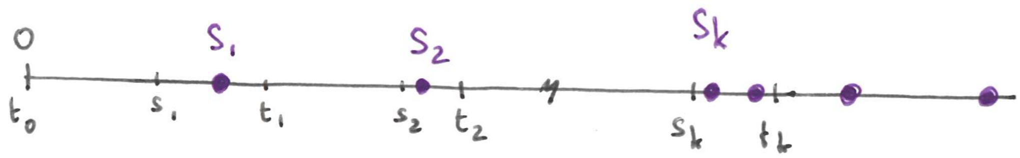
Let $U = \{(s_1, \dots, s_k) \in \mathbb{R}^k : 0 \leq s_1 \leq \dots \leq s_k\}$. We show that

$$\forall H \in \mathcal{B}(U) \quad \mathbb{P}[(S_1, \dots, S_k) \in H] = \int_H \lambda^k e^{-\lambda y_k} dy_1 \dots dy_k .$$

By Dynkin's lemma, it suffices to prove it for

$$H = [s_1, t_1) \times \dots \times [s_k, t_k),$$

with $s_1 < t_1 < \dots < s_k < t_k$. (convention: $t_0 = 0$)



$$\mathbb{P}[\forall i \leq k \quad S_i \in [s_i, t_i)]$$

$$= \mathbb{P}\left[\bigcap_{i \leq k} \{N_{s_i} - N_{t_{i-1}} = 0\} \cap \bigcap_{i < k} \{N_{t_i} - N_{s_i} = 1\} \cap \{N_{t_k} - N_{s_k} \geq 1\}\right]$$

$$= \prod_{i \leq k} e^{-\lambda(s_i - t_{i-1})} \prod_{i < k} \lambda(t_i - s_i) e^{-\lambda(t_i - s_i)} \times \left(1 - e^{-\lambda(t_k - s_k)}\right)$$

$$= \prod_{i < k} \lambda(t_i - s_i) \times e^{-\lambda s_k} \times \left(1 - e^{-\lambda(t_k - s_k)}\right)$$

$$= \prod_{i < k} \int_{s_i}^{t_i} \lambda \, dy_i \times \int_{s_k}^{t_k} \lambda e^{-\lambda y_k} dy_k$$

Hence $(S_1, \dots, S_k)$ has density $f(y_1, \dots, y_k) = \lambda^k e^{-\lambda y_k} \mathbb{1}_{y_1 < \dots < y_k}$.

Define the map $h(t_1, \dots, t_k) = (t_1, t_1 + t_2, \dots, t_1 + \dots + t_k)$. This way, we have $(T_1, \dots, T_k) = h^{-1}((S_1, \dots, S_k))$. By change of variables (and using that the Jacobian of h is 1), $(T_1, \dots, T_k)$ admits the density

$$(f \circ h)(t_1, \dots, t_k) = \lambda^k e^{-\lambda(t_1 + \dots + t_k)} \mathbb{1}_{t_1 < t_1 + t_2 < \dots < t_1 + \dots + t_k}$$

$$= \prod_{i=1}^{k} \left(\lambda e^{-\lambda t_i} \mathbb{1}_{t_i > 0} \right),$$

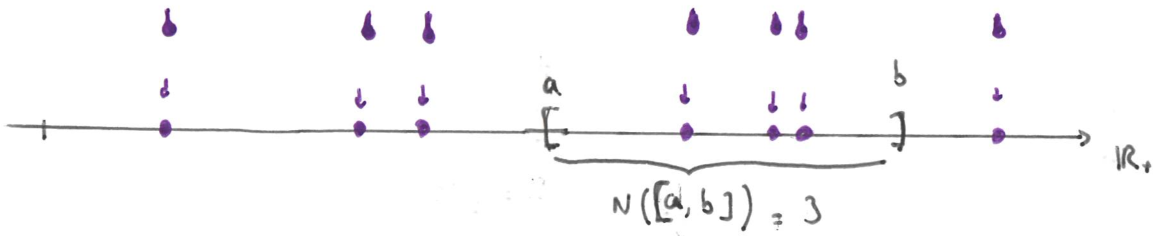
which establishes that $T_1, \dots T_k$ are iid $\mathrm{Exp}(\lambda)$

Since k is arbitrary, we conclude that the interarrival times $T_1, T_2, \dots$ are iid $\mathrm{Exp}(\lambda)$. ∎

6 MICROSCOPIC CHARACTERIZATION.

Motivation:

Droplets of water falling on the half-line $\mathbb{R}_+$.



↳ random points on $\mathbb{R}_+$ = position at which the droplets have fallen.

Define for every interval $I \subset \mathbb{R}_+$

$$N(I) = \#\{\text{random points on } [a,b]\}$$

Hypotheses:
1. If $I_1, \dots, I_k$ are disjoint intervals
   $N(I_1), \dots, N(I_k)$ are independent.

- If $I' = (a+h, b+h]$ is a translate of $I = (a,b]$, $h \geq 0$
  $$N(I)' \overset{law}{=} N(I)$$
- $\forall$ bounded interval $I \subset \mathbb{R}_+$
  $$N(I) \in \mathbb{N} \text{ a.s.}$$

The hypotheses imply that the stochastic process $N_t := N([0,t])$ is a counting process with independent and stationary increments.

We know that such process exists → Poisson process.

Q: Is it the only one ?

Answer : yes ! But we need an additional condition fixing the density λ.

Thm: Let N be a counting process, $\lambda > 0$. The following are equivalent.

(i) N is a $PP(\lambda)$

(ii) N has independent and stationary increments and

$\mathbb{P}[N_t = 1] = \lambda t + \underset{t\to 0}{o}(t)$ $\quad \left[\text{ie. } \lim_{t \to 0} \frac{P[N_t = 1]}{\lambda t} = 1 \right]$

$\mathbb{P}[N_t \geq 2] = \underset{t\to 0}{o}(t)$ $\quad \left[\text{ie } \lim_{t \to 0} \frac{\mathbb{P}[N_t \geq 2]}{t} = 0 \right]$

Lemma (Poisson approximation)

Let $(p_n)_{n\geq 1}$ be a sequence of parameters $p_n \in [0,1]$ and $\lambda \in (0,\infty)$ s.t.

$$\lim_{n\to\infty} n\, p_n = \lambda .$$

For every n, let $X_n \sim \mathrm{Bin}(n, p_n)$. Then

$$X_n \xrightarrow[n\to\infty]{} \mathrm{Poisson}(\lambda) \quad \text{in distribution.}$$

Proof: See [Probability Theory, p. 47]

Proof of Thm:

(i) $\Rightarrow$ (ii) $\mathbb{P}[N_t = 1] = \lambda t\, e^{-\lambda t} \underset{t\to 0}{=} \lambda t + o(t)$

$\mathbb{P}[N_t \geq 2] = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \underset{t\to 0}{=} o(t)$

(ii) $\Rightarrow$ (i) We already have that (N_t) has independent increments. It suffices to prove that

$$\forall\, s < t \quad N_t - N_s \sim \mathrm{Poisson}(\lambda(t-s)).$$

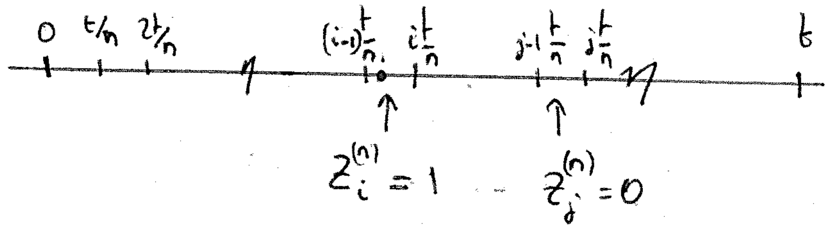
Since N_t has stationary increments, it suffices to prove

$$\forall t \quad N_t \sim \mathrm{Poisson}(\lambda t).$$

Fix $t \in (0,\infty)$.

Let $n \geq 1$. By independence and stationarity of the increments, the variables $Z_i^{(n)} = \mathbb{1}_{N_{i\frac{t}{n}} - N_{(i-1)\frac{t}{n}} \geq 1}$

are i.i.d Bernoulli (p_n) random variables, where $p_n := \mathbb{P}[N_{\frac{t}{n}} \geq 1] = \lambda \frac{t}{n} + \underset{n\to\infty}{o}\left(\frac{1}{n}\right)$



Hence $X_n := \sum_{i=1}^{n} Z_i^{(n)}$ is a Binomial (n, p_n) random variable. Since $n p_n \xrightarrow[n\to\infty]{} \lambda t$, the lemma implies for any $k \in \mathbb{N}$

$$\mathbb{P}[X_n = k] \xrightarrow[n\to\infty]{} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

We have: for every $n \geq 1$

$$\mathbb{P}[N_t \neq X_n] = \mathbb{P}\Big[\bigcup_{1 \leq i \leq n} \{N_{i\frac{t}{n}} - N_{(i-1)\frac{t}{n}} \geq 2\}\Big]$$

$$\overset{\text{union bound}}{\leq} \sum_{i=1}^{n} \mathbb{P}\left[N_{i\frac{t}{n}} - N_{(i-1)\frac{t}{n}} \geq 2\right]$$

$$\overset{\text{stationarity}}{=} n \times \mathbb{P}\left[N_{\frac{t}{n}} \geq 2\right]$$

Since $\mathbb{P}[N_{t/n} \geq 2] = \underset{n\to\infty}{o}\left(\frac{t}{n}\right)$, we get that

$$\lim_{n\to\infty} \mathbb{P}[N_t \neq X_n] = 0.$$

Fix $k \in \mathbb{N}$. For every $n \geq 1$

$$|\mathbb{P}[N_t = k] - \mathbb{P}[X_n = k]| \leq \mathbb{E}\Big[\underbrace{|\mathbb{1}_{N_t = k} - \mathbb{1}_{X_n = k}|}_{\leq \mathbb{1}_{N_t \neq X_n}}\Big] \leq \mathbb{P}[N_t \neq X_n].$$

Hence $\mathbb{P}[N_t = k] = \lim_{n\to\infty} \mathbb{P}[X_n = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. ∎

4 PROPERTIES OF THE POISSON PROCESSES

4.1 Law of large numbers

Thm: Let $(N_t)_{t\geq 0}$ be a Poisson process on $(\Omega, \mathcal{F}, P)$ of rate λ.

Then we have

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \quad \text{a.s.}$$

Proof: Let $X_i = N_i - N_{i-1}$, $i \geq 1$.

The X_i are pairwise independent, identically distributed random variables (actually, they are i.i.d.)

$X_1 \sim \text{Poisson}(\lambda)$. Hence

$$E[|X_1|] = E[X_1] = \lambda < \infty .$$

By the strong law of large number (see [PROBABILITY, p.36])

$$\frac{X_1 + \dots + X_i}{i} \xrightarrow[i\to\infty]{} \lambda \quad \text{a.s.}$$

Since for every $i \leq t < i+1$

$$\frac{X_1 + \dots + X_i}{i+1} \leq \frac{N_t}{t} \leq \frac{X_1 + \dots + X_{i+1}}{i}$$

We also have $\frac{N_t}{t} \xrightarrow[t\to\infty]{} \lambda$ a.s. ∎

4.2 Thining

Let $(N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda > 0$, jump times (S_k).

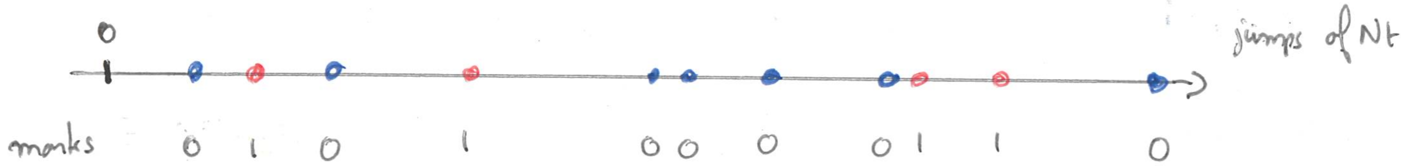
Let $(X_n)_{n \geq 1}$ i.i.d Bernoulli variable with parameter $p \in (0,1)$

"such a sequence is called a marking of $(N_t)_{t \geq 0}$".

Define the thined processes.

$$N_t^1 = \sum_{k \geq 1} \mathbb{1}_{\{S_k \leq t,\, X_k = 1\}}$$

$$N_t^0 = \sum_{k \geq 1} \mathbb{1}_{\{S_k \leq t,\, X_k = 0\}}.$$



Applications: • Clients of type 0 and type 1 in a queue.

• Valid / non valid claims at insurance Compagny.

Rk: $N_t = N_t^0 + N_t^1$ a.s.

Theorem: (thining of Poisson processes)

$$\boxed{(N_t^0)_{t \geq 0} \text{ and } (N_t^1)_{t \geq 0} \text{ are independent Poisson processes with rate } \lambda_0 = (1-p)\lambda \text{ and } \lambda_1 = p\lambda \text{ resp.}}$$

Proof: Later with punctual Poisson processes on general spaces "just families of random variables"

Rk Let $(X_t)_{t \geq 0}$, $(Y_t)_{t \geq 0}$ be two stochastic processes on the same probability space. They are *independent* iff

$$\forall t_1 < \dots < t_k \quad \forall s_1 < \dots < s_l \quad (X_{t_1}, \dots, X_{t_k}) \text{ indep. of } (Y_{s_1}, \dots, Y_{s_l}).$$

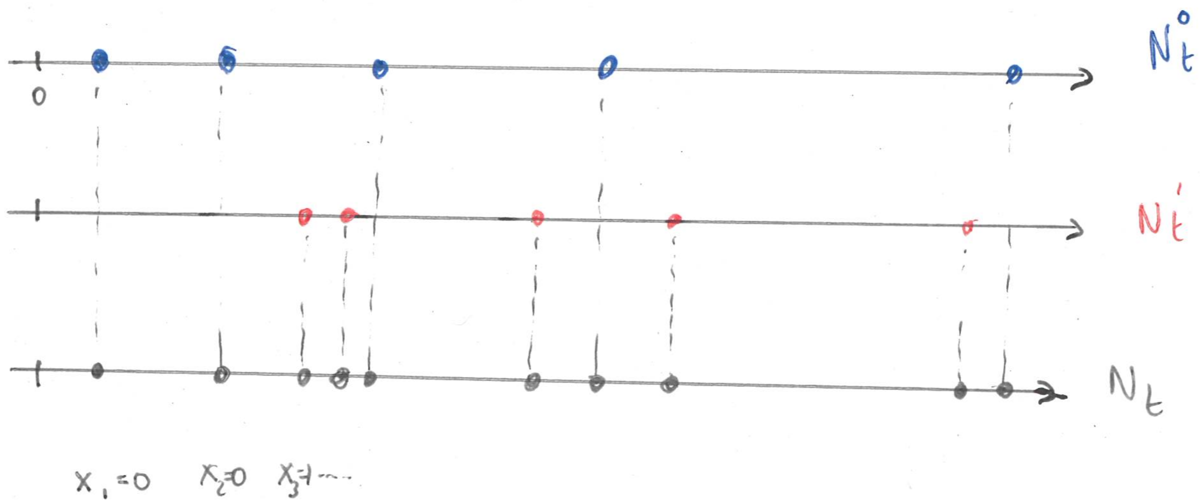
"random vectors"

4.3 Superposition.

Let $(N_t^0)_{t \geq 0}$ and $(N_t^1)_{t \geq 0}$ be two independent Poisson processes with respective rates $\lambda_0 > 0$, $\lambda_1 > 0$.

Define

$$N_t = N_t^0 + N_t^1 \quad , \; t \geq 0. \qquad \text{"superposition process"}$$



N_t is a counting process and we define

$$X_k = \mathbb{1}\left\{\begin{array}{l}\text{the } k\text{-th jump of } N_t \text{ is} \\ \text{a jumping time of } N_t^1\end{array}\right\}$$

Thm: (superposition of Poisson processes)

$(N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda = \lambda_0 + \lambda_1$ and $(X_k)_{k\geq 1}$ is a marking of $(N_t)_{t\geq 0}$ with

$$\forall k \quad P[X_k = 1] = \frac{\lambda_1}{\lambda_0 + \lambda_1}.$$

Proof: $(N_t)_t$ is a counting process (it follows from the definition.)

We consider (independently of N^0, N^1)

- $(\tilde{N}_t)_{t\geq 0}$ Poisson process intensity $\lambda = \lambda_0 + \lambda_1$
- $(\tilde{X}_k)_{k\geq 1}$ iid Bernoulli $\left(\frac{\lambda_1}{\lambda_0+\lambda_1}\right)$.

By the theorem in the previous section, the thined processes $(\tilde{N}_t^0), (\tilde{N}_t^1)$ are independent processes with respective rates λ_0, λ_1.

For every $t_1 < \dots < t_k$, $f : \mathbb{R}^k \to \mathbb{R}$ bounded.

$$\begin{aligned} E[f(N_{t_1}, \dots, N_{t_k})] &= E[f(N_{t_1}^0 + N_{t_1}^1, \dots, N_{t_k}^0 + N_{t_k}^1)] \\ &= E[f(\tilde{N}_{t_1}^0 + \tilde{N}_{t_1}^1, \dots, \tilde{N}_{t_k}^0 + \tilde{N}_{t_k}^1)] \\ &= E[f(\tilde{N}_{t_1}, \dots, \tilde{N}_{t_k})] \end{aligned}$$

Therefore N is a $PP(\lambda)$. Similarly, for every $t_1 < \dots < t_k$ for every $\ell \geqslant 1$ and every $f : \mathbb{R}^k \times \{0,1\}^\ell \to \mathbb{R}$ meas. bounded

$$\mathbb{E}\left[f(N_{t_1}, \dots, N_{t_k}, X_1, \dots, X_\ell)\right] = \mathbb{E}\left[f(\tilde{N}_{t_1}, \dots, \tilde{N}_{t_k}, \tilde{X}_1, \dots \tilde{X}_k)\right]$$

Hence $X_1, \dots, X_\ell$ are iid Bernoulli$\left(\frac{\lambda_1}{\lambda_0 + \lambda_1}\right)$ indep. of $(N_{t_1}, \dots, N_{t_k})$. ■