CHAPTER 4

GENERAL POISSON POINT PROCESSES.

Ref: [LAST - PENROSE] Lectures on the Poisson process.

[KINGMAN] Poisson processes.

INTRODUCTION.

How to represent points on $\mathbb{R}^+$ mathematically?

1. time viewpoint $\rightsquigarrow T_1, T_2, \dots$ $\qquad T_i$ = "time between $(i-1)$th point and ith point"

2. càdlàg process with values in $\mathbb{N}$ $\qquad N_t$ = "number of points in $[0,t]$"

3. set of points $\varphi = \{S_1, S_2, \dots\} \subset \mathbb{R}_+$.

4. measure : $N : \mathcal{B}(\mathbb{R}_+) \longrightarrow \mathbb{N}$
$$A \longmapsto \text{"number of points in } A\text{"}$$

1, 2 are very specific to 1-dimension.

3 is general but the description $S_1, S_2, \dots$ require a choice ; what is the first point?

4 is a bit abstract but is very convenient to work with.

Goal: define Poisson processes on more general space than $\mathbb{R}_+$.

- $(\Omega, \mathcal{F}, \mathbb{P})$ abstract probability space

Framework: • (E, d) Polish space (separable, complete metric space).

$\to$ $\mathcal{E}$ = Borel σ-algebra.

(Notation: $B_i \uparrow E$ if $B_1 \subset B_2 \subset \dots$ meas. and $E = \bigcup_{i \geq 1} B_i$)

- μ σ-finite measure

$\to$ $\exists B_i \uparrow E$ s.t. $\forall i$ $\mu(B_i) < \infty$

Examples: • $E = \mathbb{R}_+$ $\quad \mu = \lambda \cdot \mathrm{Leb}_{\mathbb{R}_+}$

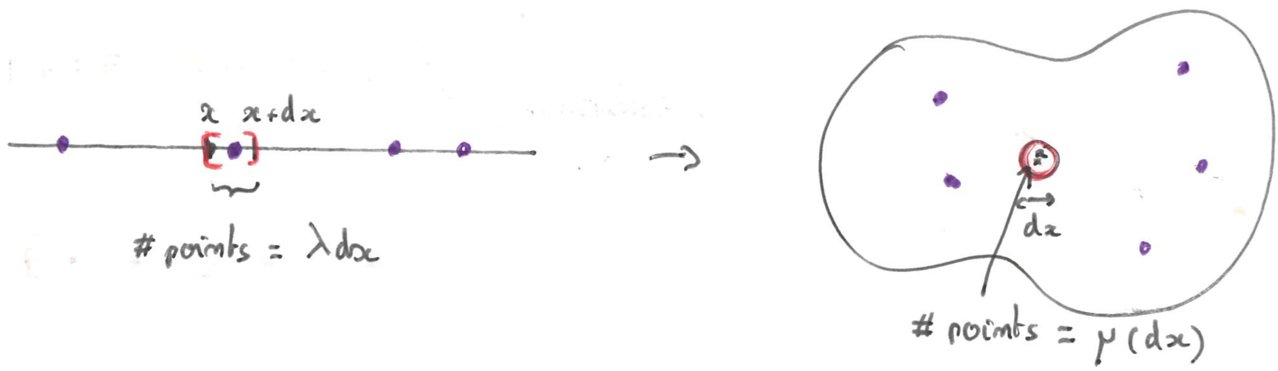
• $E = \mathbb{R}^2$ $\quad \mu = \lambda\, \mathrm{Leb}_{\mathbb{R}^2}$

• $E = \mathbb{R}^2$ $\quad \mu(dx) = \frac{1}{\pi} e^{-|x|^2} dx$ "Gaussian"

? we wish to define a point process on $(E, \mathcal{E})$ where

"number of points around x" $\simeq \mu(dx)$

on $\mathbb{R}_+ = [0, \infty)$ $\qquad\qquad$ on $(E, \mathcal{E})$



I POINT PROCESSES.

Notation: $\mathcal{N} := \{\sigma\text{-finite measures s.t. } \forall B \in \mathcal{E} \quad \sigma(B) \in \mathbb{N} \cup \{+\infty\}\}$

$\mathcal{X} \subset E$ "σ-finite" $\longrightarrow$ $\eta \in \mathcal{N}$

$(\exists B_i \uparrow E : |\mathcal{X} \cap B_i| \in \mathbb{N})$ $\quad$ $(\eta(B) := |\mathcal{X} \cap B|)$

Example $E = \mathbb{R}$, $\eta = \sum_{x \in \mathbb{Z}} \delta_x$, $\eta = \sum_{i=1}^{\infty} \delta_{1/i}$.

measured structure:

Let $\mathcal{B}(\mathcal{N})$ be the σ-algebra generated by the sets

$\{\eta : \eta(B) = k\}$, $B \subset E$ meas., $k \in \mathbb{N}$.

Prop. [Representation as a Dirac sum]

Let $\mathcal{N}_{<\infty} = \{\eta \in \mathcal{N} : \eta(E) < \infty\}$. There exist measurable maps $\tau : \mathcal{N}_{<\infty} \longrightarrow \mathbb{N}$ and $X_i : \mathcal{N}_{<\infty} \longrightarrow E$. Such that

$$\forall \eta \in \mathcal{N}_{<\infty} \quad \eta = \sum_{i=1}^{\tau(\eta)} \delta_{X_i(\eta)}$$

$\eta \longleftrightarrow \{X_1, \dots, X_\tau\}$

Rk: multiple points are possible.

Proof: Fix $\mathcal{Y} = \{y_1, y_2, \dots\}$ countable dense in E.

(WLOG we can assume that E is infinite).

We have $\mathcal{N} = \bigcup_{k=0}^{\infty} \mathcal{N}_k$ (disjoint) where $\mathcal{N}_k = \{\eta : \eta(E) = k\}$

We prove by induction on $k \geq 0$ that

$\forall k \geq 0 \quad \exists\, Z_1, \dots, Z_k : \mathcal{N}_k \longrightarrow E$ measurable s.t

$$\forall \eta \in \mathcal{N}_k \qquad \eta = \sum_{i=1}^{k} \delta_{Z_i}$$

- For $k = 0$, there is nothing to prove.
- Let $k \geq 0$ and assume that the property holds.

Let $\eta \in \mathcal{N}$ s.t $\eta(E) = k+1 \; (\geq 1)$.

We will construct by induction $Y_1(\eta), Y_2(\eta) \dots \in \mathcal{Y}$

s.t $\;$ - $\forall n \quad Y_n$ is measurable (as a mapping $\{\eta : \eta(E) = k+1\} \to E$)

- $\forall n \quad \eta\left(\bigcap_{m \leq n} B\left(Y_m, \frac{1}{m}\right)\right) \geq 1$

$\hookrightarrow$ "ball in E"

<u>construction of Y_1</u>

$$\text{We have } 1 \leq \mu(E) \underset{\substack{\uparrow}}{\leq} \sum_{i \geq 1} \eta\left(B(y_i, 1)\right)$$

because $E = \bigcup_{i \geq 1} B(y_i, 1)$.

Define $i_1 := \min\{i : \eta(B(y_i, 1)) \geq 1\}$ and set $Y_1(\eta) = y_{i_1}$.

$\hookrightarrow Y_1$ is meas. because

$$\{Y_1(\eta) = y_j\} = \bigcap_{i < j} \{\eta(B(y_i, 1)) = 0\} \cap \{\eta(B(y_j, 1)) = 1\}.$$

- Construction of Y_n assuming $Y_1, \dots, Y_{n-1}$ constructed.

Let $C := \bigcap_{1 \le m \le n-1} B\left(Y_m(\eta), \frac{1}{m}\right)$. We have

$$1 \le \eta(C) \le \sum_{i \ge 1} \eta\left(C \cap B\left(y_i, \frac{1}{n}\right)\right)$$

Define $Y_n(\eta) = y_{i_n}$ where $i_n = \min\{i : \eta(C \cap B(y_i, \frac{1}{n})) \ge 1\}$.

As above Y_n is measurable.

The sequence $(Y_n)_{n \ge 0}$ constructed above is a Cauchy sequence (Indeed, for every $n \ge m$ $B(Y_n, \frac{1}{n}) \cap B(Y_m, \frac{1}{m}) \ne \emptyset$ hence by the triangle inequality $d(Y_n, Y_m) \le \frac{2}{m}$)

Define $Z_{k+1}(\eta) = \lim_{n \to \infty} Y_n(\eta)$ (Z_{k+1} is measurable as a simple limit of measurable functions).

Furthermore $\{Z_{k+1}(\eta)\} = \bigcap_{n \ge 1} B\left(Y_n, \frac{2}{n}\right)$ and therefore $\eta(\{Z_{k+1}(\eta)\}) \ge 1$.

Define $\eta' = \eta - \delta_{Z_{k+1}(\eta)}$ (η' is measurable in η)
$\eta'(E) = k$. By induction, there exist $\tilde{Z}_1(\eta'), \dots, \tilde{Z}_k(\eta')$ s.t. $\eta' = \delta_{\tilde{Z}_1} + \dots + \delta_{\tilde{Z}_k}$. Defining $Z_i(\eta) = \tilde{Z}_i(\eta')$, we obtain

$$\eta = \sum_{i=1}^{k+1} \delta_{Z_i(\eta)}.$$

■

Def: Let $(\Omega, \mathcal{F}, P)$ be a probability space

$$\boxed{\text{A point process on } (E, \mathcal{E}) \text{ is a random variable } N \text{ defined on } \Omega \text{ with values in } \mathcal{N}.}$$

→ "N is a random measure"

This means:

$$\begin{aligned} N : \Omega &\longrightarrow \mathcal{N} \qquad \text{measurable} \\ \omega &\longmapsto N_\omega \end{aligned}$$

For fixed $B \subset E$ measurable, one can consider

$$\begin{aligned} N(B) : \Omega &\longrightarrow \mathbb{N} \cup \{+\infty\} \\ \omega &\longmapsto N_\omega(B) \end{aligned}$$

and one can directly check that $N(B)$ is a random variable.

$N(B)$ = "(random) number of points in B"

Examples

- $N = 0$ a.s. → "empty set."
- $N = \delta_X$ where $E = [0,1]$ and X r.v. in $[0,1]$.
- $N = \delta_{X_1} + \dots + \delta_{X_n}$ "Bernoulli process"

  where $X_1, \dots, X_n$ r.v. with values in E
- $N = \delta_{\lambda_1} + \dots + \delta_{\lambda_n}$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a $n \times n$ Wigner matrix.

2 POISSON POINT PROCESSES

Def: A Poisson process with intensity measure μ $(\text{p.p.p.}(\mu))$ on $(E, \mathcal{E})$ is a p.p. such that

(i) $\forall B_1, \dots, B_k \subset E$ measurable disjoint

$$N(B_1), \dots, N(B_k) \text{ are independent.}$$

(ii) $\forall B \subset E$ measurable

$$N(B) \sim \text{Poisson}(\mu(B)).$$

Rk: $\forall B \in E$ measurable

$$E[N(B)] = \mu(B) \rightarrow \text{"in average, there are } \mu(B) \text{ points in } B\text{"}.$$

Ex1 $E = [0, \infty)$, $\mathcal{E} = \mathcal{B}([0,\infty))$

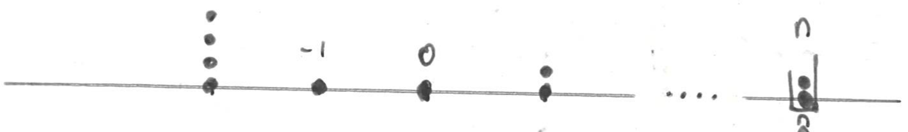
Let $(N_t)_{t \geq 0}$ be a standard Poisson process (intensity $\lambda > 0$) with jump times $S_1, S_2, \dots$

Then $N = \sum_{i=1}^{\infty} \delta_{S_i}$ is a ppp $(\lambda . \text{Leb}_{[0,\infty)})$

Ex2 $E = \mathbb{Z}$, $\mathcal{E} = \mathcal{P}(\mathbb{Z})$

Let $(X_n)_{n \in \mathbb{Z}^2}$ be iid $X_n \sim \text{Poisson}(\lambda)$

Then $N = \sum_{n \in \mathbb{Z}} X_n \delta_n$ ppp. $(\lambda . \underbrace{|\cdot|}_{\text{"counting measure"}})$



X_n points at n

Thm: [representation as a proper process]

Let N be a ppp (μ) on $(E, \mathcal{E})$. There exist some r.v. $\tau \in \mathbb{N} \cup \{+\infty\}$ and $X_n \in E, n \geq 1$ defined on Ω s.t.

$$N = \sum_{n=1}^{\tau} \delta_{X_n}$$

Proof. Let $B_i \uparrow E$ s.t. $\mu(B_n) < \infty$. Set $A_i = B_i \setminus B_{i-1}, n \geq 0$ (A_n are disjoint and their union is E).

The process $N_i := N(\cdot \cap A_i)$ takes values on $\mathcal{N}_{<\infty}$.

Hence the proposition in the previous section ensures that there exist some r.v. $\tau^{(i)}, Z_1^{(i)}, \dots, Z_\tau^{(i)}$ s.t.

$$N_i = \sum_{j=1}^{\tau^{(i)}} \delta_{Z_j^{(i)}} \quad \text{a.s.}$$

Using that $N = \sum_{i=1}^{\infty} N_i$, and a reordering of the terms in the sums, we obtain the desired result ■

Question: Given μ, does there always exist a ppp(μ)?

3 EXISTENCE AND UNIQUENESS

3.1 SPACE WITH FINITE MEASURE

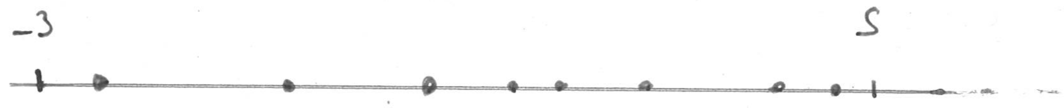
Assume $\mu(E) < \infty$.

Prop: Let $Z, X_i, i \geq 1$ independent random variables

$$Z \sim \text{Poisson}(\mu(E)), \quad X_i \sim \frac{\mu[\cdot]}{\mu[E]}. \quad \leftarrow \text{"} P(X_i \in A) = \frac{\mu(A)}{\mu(E)} \text{"}$$

Then $N = \sum_{i=1}^{Z} \delta_{X_i}$ is a $PPP(\mu)$

<u>concrete example</u> $E = [-3, 5]$. $\mu = \text{Leb}$.



$\leftarrow Z$ uniform points in $[-3,5)$

$$Z \sim \text{Poisson}(8) \qquad X_i \sim \frac{1}{8}\,\text{Leb}_{[-3,5]}[\cdot] = \mathcal{U}([-3,5])$$

$$N = \sum_{i=1}^{Z} \delta_{X_i}.$$

Proof: Let $B_1, \dots, B_{k-1} \subset E$ disjoint measurable. Set $B_k := E \setminus B_1 \cup \dots \cup B_k$

Let $n = n_1 + n_2 + \dots + n_k$, Define $Y_i = \underbrace{\sum_{j=1}^{n} \mathbb{1}_{X_j \in B_i}}_{\text{"number of } X_j\text{'s in } B_i\text{"}}$

Observe that

$$(Y_1, \dots, Y_k) \sim \text{multinomial}\left(\frac{\mu(B_1)}{\mu(E)}, \dots, \frac{\mu(B_k)}{\mu(E)}\right)$$

independent of Z.

We have

$$P[N(B_1)=n_1,\dots,N(B_k)=n_k]$$

$$= P[Z=n\ ,\ Y_1=n_1,\dots,Y_k=n_k]$$

$$\overset{\text{indep.}}{=} \frac{\mu(E)^n}{n!}e^{-\mu(E)} \times \frac{n!}{n_1!\cdots n_k!}\left(\frac{\mu(B_1)}{\mu(E)}\right)^{n_1}\cdots\left(\frac{\mu(B_k)}{\mu(E)}\right)^{n_k}$$

$$= \prod_{i=1}^{k}\frac{\mu(B_i)^{n_i}}{n_i!}e^{-\mu(B_i)}.$$

By summing over all n_k; we get

$$P[N(B_1)=n_1,\dots,N(B_{k-1})=n_{k-1}] = \prod_{i=1}^{k-1}\frac{\mu(B_i)^{n_i}}{n_i!}e^{-\mu(B_i)}$$

Hence $N(B_1),\dots,N(B_{k-1})$ are ind. and $N(B_i)\sim$ Poisson$(\mu(B_i))$ ∎

3.2 SUPERPOSITION AND EXISTENCE.

Lemma: Let $\lambda = \sum_{i=1}^{\infty}\lambda_i \quad \lambda_i \geq 0$

Let $X_i \sim$ Poisson(λ_i), $i\geq 1$ independent.

Then $X = \sum_{i=1}^{\infty} X_i \sim$ Poisson(λ)

Convention $\quad X\sim$ Poisson$(\infty) \Leftrightarrow X=+\infty$ a.s.

Proof: see exercises.

Thm (Superposition):

> Let $N_i, i \geq 1$ be a sequence of independent $PPP(\mu_i)$ where μ_i and $\mu := \sum_{i=1}^{\infty} \mu_i$ σ-finite measures
> Then $N = \sum_{i=1}^{\infty} N_i$ is a $PPP(\mu)$ where $\mu = \sum_i \mu_i$.

Proof. We first check that N is a p.p.

$\rightarrow$ Let $B_n \uparrow E$ s.t. $\mu(B_n) < \infty$.

$$\forall n \quad N(B_n) = \sum_{i=1}^{\infty} \underbrace{N_i(B_n)}_{\sim \text{Poisson}(\mu_i(B_n))}$$

Since $\mu(B_n) < \infty$ $\quad N(B_n) < \infty$ a.s.

Hence N is a σ-finite measure a.s.

Furthermore $\forall$ $B \subset E$ measurable $N(B) = \sum_i N_i(B)$ measurable
Hence N is measurable.

For $B \subset E$ measurable $N(B) = \sum_i \underbrace{N_i(B)}_{\sim \text{Poisson}(\mu_i(B))}$

By the lemma, $N(B) \sim \text{Poisson}(\mu(B))$

Finally for $B_1, \dots, B_k \subset E$ meas. disjoint $(N_i(B_j))_{i \in \mathbb{N}, 1 \leq j \leq k}$ are ind. r.v. Therefore

$N(B_1) = \sum_i N_i(B_1)$, $\dots$, $N(B_k) = \sum_i N_i(B_k)$ are indep., by grouping. ∎

Corollary.

> Assume that μ is a σ-finite measure on $(E, \mathcal{E})$.
> Then there exists a PPP (μ) on E.

Proof: $\mu = \sum_{i=1}^{\infty} \mu_i$ where $\mu_i(E) < \infty$.

Let (N_i) independent Poisson processes, where N_i PPP(μ_i).

Define $N = \sum_{i=1}^{\infty} N_i$. By superposition, N is a PPP(μ) ∎

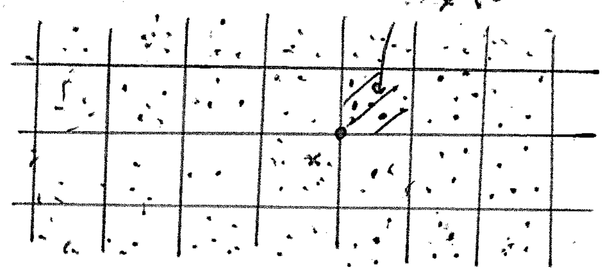
A concrete case : PPP $(\lambda . \mathrm{Leb}_{\mathbb{R}^2})$

$E = \mathbb{R}^2 \quad \mathcal{E} = \mathcal{B}(\mathbb{R}^2) \quad \mu = \lambda . \mathrm{Leb}_{\mathbb{R}^2}, \ \lambda > 0$.

Define for $x \in \mathbb{Z}^2 \quad x = (x_1, x_2)$

$$\mu_x = \lambda . \mathrm{Leb}_{[x_1, x_1+1) \times [x_2, x_2+1)}$$

Z_x points



Then $\mu = \sum_{x \in \mathbb{Z}^2} \mu_x$.

Consider independent r.v. $Z_x, x \in \mathbb{Z}^2$ $U_{x,i}, x \in \mathbb{Z}^2, i \in \mathbb{N}$

$Z_x \sim \text{Poisson}(\lambda) \qquad U_{x,i} \sim \mathcal{U}\left([x_1, x_1+1) \times [x_2, x_2+1)\right)$

Then $N = \sum_{x \in \mathbb{Z}^2} \sum_{i=1}^{Z_x} \delta_{U_{x,i}}$ PPP $(\lambda . \mathrm{Leb})$

3.3 UNIQUENESS

Let N be a ppp (μ) on E. Its law P_N is a probability measure P_N on $\mathcal{M}$.

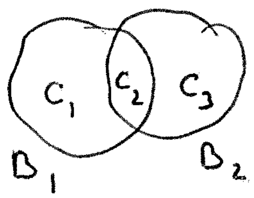
> **Prop.** Let N, N' be two p.p.p (μ) on $(E, \mathcal{E})$.
> Then $P_N = P_{N'}$.

<u>Rk</u>: $P_N = P_{N'} \iff \forall A \subset \mathcal{M}$ measurable $P_N(A) = P_{N'}(A)$

$\iff \forall A \subset \mathcal{M}$ measurable $\mathbb{P}[N \in A] = \mathbb{P}[N' \in A]$

"probability measure on the underlying probability space"

<u>Proof</u>: Let $B_1, B_2 \subset E$ measurable, $n_1, n_2 \geq 0$



$C_1 = B_1 \cap B_2^c$
$C_2 = B_1 \cap B_2$
$C_3 = B_1^c \cap B_2$

$$P[N(B_1) = n_1, N(B_2) = n_2] = \sum_{\substack{m_1 + m_2 = n_1 \\ m_2 + m_3 = n_2}} P[N(C_1) = m_1, N(C_2) = m_2, N(C_3) = m_3]$$

$$\overset{C_1, C_2, C_3 \text{ disj}}{=} \sum_{\substack{m_1 + m_2 = n_1 \\ m_2 + m_3 = n_2}} P[N'(C_1) = m_1, N'(C_2) = m_2, N'(C_3) = m_3]$$

$$= P[N'(B_1) = n_1, N'(B_2) = n_2]$$

Equivalently $\forall B_1, \dots, B_k \subset E$ measurable

$$P[N(B_1) = n_1, \dots N(B_k) = n_k] = P[N'(B_1) = n_1, \dots, N'(B_k) = n_k]$$

(Hint: consider a partition of E with sets of the form

$$C = B_1^{\circ} \cap \dots B_k^{\circ} \quad \text{where } B_i^{\circ} = B_i \text{ or } B_i^c.)$$

Therefore
$$P_N(A) = P_{N'}(A) \qquad (*)$$

for every set of the form

$$A = \{\eta : (\eta(B_1), \dots, \eta(B_k)) \in K\} \quad B_1, \dots, B_k \subset E \text{ measurable}, \; K \subset \mathbb{N}^k.$$

Such sets form a π-system and generate $\mathcal{B}(\mathcal{N})$
Hence, by Dynkin's lemma, $(*)$ holds for every measurable set $A \subset \mathcal{N}$. ∎

4 LAPLACE FUNCTIONAL

Lemma: Let $X \sim \text{Poiss}(\lambda)$ $\quad \lambda > 0$
Then $\forall u \geq 0$ $\quad E[e^{-uX}] = \exp[-\lambda(1-e^{-u})]$

Proof: $E[e^{-uX}] = \sum_k \frac{\lambda^k}{k!} e^{-\lambda} e^{-ku} = e^{-\lambda} \times \exp(\lambda e^{-u})$ ∎

Def: (Laplace functional for p.p.)
Let N p.p. on $(E, \mathcal{E})$. For every $u : E \to \mathbb{R}_+$ measurable, define

$$\boxed{L_N(u) := \mathbb{E}\left[\exp\left(-\int u(x)\, N(dx)\right)\right]}$$

<u>Rk</u>: $L_N(u)$ is well defined.

Indeed $\int u(x)\, N(dx)$ is a well defined random variable (see exercises).

<u>Interpretation of $\int u(x)\, N(dx)$</u>

$$\int u(x)\, N(dx) = \sum_{\substack{x \text{ "points of } N\text{"} \\ \text{counted with} \\ \text{multiplicity}}} u(x)$$

<u>Thm</u>: [Characterization of Poisson process via Laplace functional.]

Let μ σ-finite measure on $(E, \mathcal{E})$. Let N p.p. on E.
The following are equivalent

(i) N is a PPP(μ)

(ii) $\forall u : E \longrightarrow \mathbb{R}^+$ measurable

$$L_N(u) = \exp\left(-\int (1-e^{-u(x)})\, \mu(dx)\right)$$

<u>Proof</u>: Let $u = \sum_{i=1}^{k} u_i \mathbb{1}_{B_i}$ $\quad B_1, \dots, B_k$ disj $\quad u_i \geq 0$.

$$L_N(u) = E\left[\exp\left(-\sum_{i=1}^{k} u_i N(B_i)\right)\right] \overset{\text{ind.}}{=} \prod_{i=1}^{k} E\left[e^{-u_i N(B_i)}\right]$$

$$\overset{\text{lemma}}{=} \prod_{i=1}^{k} \exp\left(-\mu(B_i)(1-e^{-u_i})\right) = \exp\left[-\int_E 1-e^{-u(x)}\, \mu(dx)\right]$$

For general $u \geq 0$, consider (u_n) of the form above such that $u_n \uparrow u$. For every n

$$\underbrace{L_N(u_n)}_{\substack{\downarrow n\to\infty \\ L_N(u)}} = \underbrace{\exp\left(-\int_E (1-e^{-u_n(x)})\mu(dx)\right)}_{\substack{\downarrow n\to\infty \\ \exp\left(-\int_E (1-e^{-u(x)})\mu(dx)\right)}}$$

monotone cv

$\Leftarrow$ Let $B_1, \dots, B_k$ disj. $\forall x = (x_1, \dots, x_k)$ $x_i \geq 0$,

$$E[e^{-x\cdot(N(B_1),\dots,N(B_k))}] \overset{\text{by def}}{=} L_N(u) \quad \text{where } u = \sum x_i \mathbb{1}_{B_i}$$

$$\overset{\text{assumption}}{=} \exp\left[-\int_E 1-e^{-u(x)}\,\mu(dx)\right]$$

$$= \prod_{i=1}^k \exp\left(-\mu(B_i)(1-e^{-x_i})\right)$$

$$\underset{\substack{\uparrow \\ \text{as previous computation}}}{=} E\left[e^{-x\cdot Y}\right]$$

where $Y = (Y_1, \dots, Y_k)$ rand vector of independant variables

$$Y_i \sim \text{Poisson}(\mu(B_i))$$

Since the Laplace transform characterizes the law, we have

$$(N(B_1),\dots,N(B_k)) \overset{\text{law}}{=} Y$$

$\blacksquare$

5 MAPPING:

$(E, \mathcal{E})$, $(F, \mathcal{F})$ Polish spaces. μ σ-finite measure on E.

Thm [Mapping] $T: E \to F$ measurable.

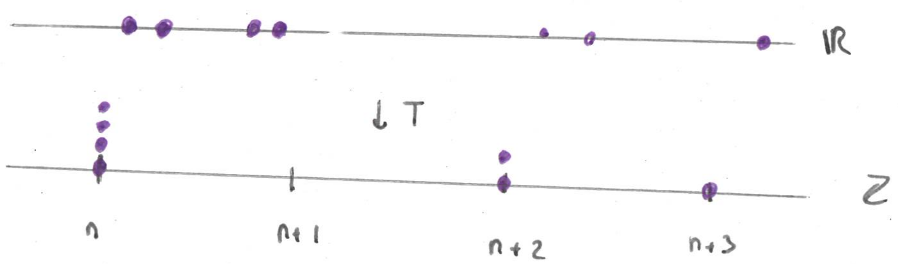
> Assume that the pushforward measure $T\#\mu$ (def. by $T\#\mu(B) = \mu(T^{-1}(B))$) on F is σ-finite. Let N be a PPP(μ) on E. Then $T\#N$ is a PPP ($T\#\mu$) on F.

Pf: see exercises.

Rk: If N is proper $N = \sum_{i=1}^{\zeta} \delta_{x_i}$, then $T\#N$ is also proper and $T\#N = \sum_{i=1}^{\zeta} \delta_{T(x_i)}$.

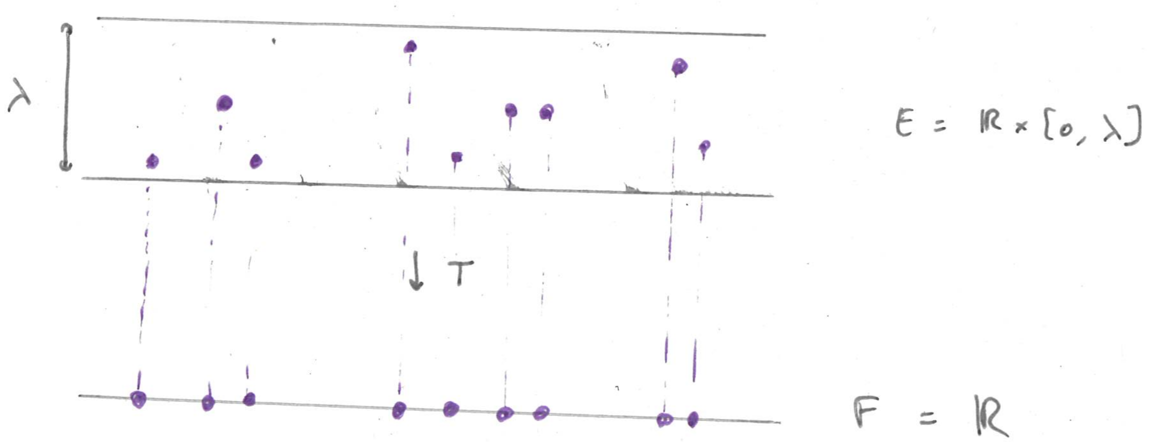
Ex1 $E = \mathbb{R}$, $F = \mathbb{Z}$ $\quad T: E \longrightarrow F$, $x \longmapsto \lfloor x \rfloor$

If N PPP(Leb) on $\mathbb{R}$, then $T\#N$ PPP($|\cdot|$) on $\mathbb{Z}$.



Ex2 $E = \mathbb{R} \times [0, \lambda)$ $\mu = $ Leb $\quad F = \mathbb{R}$ $\quad T: (x_1, x_2) \longmapsto x_1$

If N PPP(Leb) on E, then $T\#N$ PPP(λ.Leb) on $\mathbb{R}$



Ex 3. If $T : \mathbb{R}^d \to \mathbb{R}^d$ $\mathcal{C}^1$-diffeomorphism

If N PPP(Leb) on $\mathbb{R}^d$, then $T\# N$ PPP($h \cdot$ Leb)

where $h(x) = \det(dT^{-1}(x))$ jacobian.

→ "generalization of the relation homogeneous / inhomogeneous Poisson processes."

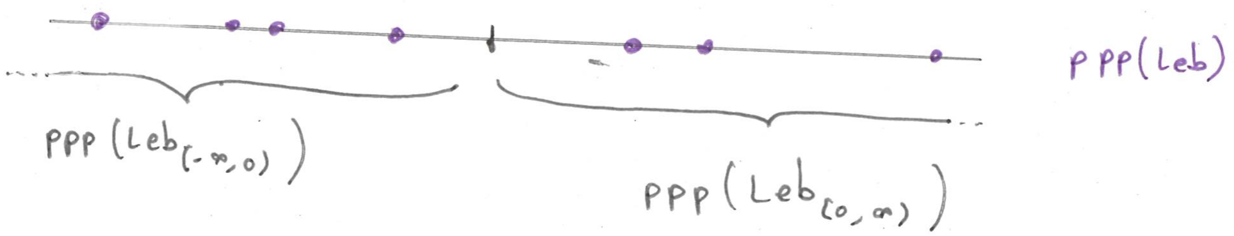
6 RESTRICTION.

Not: If ν measure on E, $C \subset E$ measurable, $\nu_C := \nu(\cdot \cap C)$ "measure restricted to C".

Thm [restriction]

> $(E, \mathcal{E})$ Polish space, μ σ-finite. $C_1, C_2, \dots \subset E$ meas. disjoint.
> If N PPP(μ) on E, then $N_{C_1}, N_{C_2}, \dots$ are indep. PPP with respective intensity measures $\mu_{C_1}, \mu_{C_2}, \dots$

Ex: $E = \mathbb{R}$ $\quad C_1 = (-\infty, 0)$ $\quad C_2 = [0, \infty)$



<u>Proof</u>: Without loss of generality, we may assume

$$E = \bigcup_{i \geq 1} C_i .$$

Let $\tilde{N}_1, \tilde{N}_2, \dots$ ind. PPP with resp. intensity $\mu_{C_1}, \mu_{C_2}, \dots$

By superposition $\tilde{N} := \sum_{i \geq 1} \tilde{N}_i$ is a PPP (μ)

(indeed, $\mu = \sum_{i \geq 1} \mu_{C_i}$)

For every $B \subset E$ measurable and $j \geq 1$

$$\tilde{N}(B \cap C_j) = \sum_{i \geq 1} \underbrace{\tilde{N}_i (B \cap C_j)}_{= \begin{cases} 0 \text{ a.s if } i \neq j \\ \tilde{N}_i(B) \text{ a.s. if } i = j \end{cases}} = \tilde{N}_j (B) \text{ a.s.}$$

Hence $\tilde{N}_{C_j} = \tilde{N}_j$ a.s.

Let $f_1, \dots, f_k : \mathcal{N} \longrightarrow \mathbb{R}_+$ measurable

$$E\left[\prod_{i=1}^{k} f_i(N_{C_i})\right] \underset{\substack{\uparrow \\ \text{uniqueness} \\ (N \text{ and } \tilde{N} \text{ have the same} \\ \text{distribution})}}{=} E\left[\prod_{i=1}^{k} f_i(\tilde{N}_{C_i})\right] \underset{\substack{\uparrow \\ \tilde{N}_{C_i} = N_i \text{ a.s.}}}{=} E\left[\prod f_i(\tilde{N}_i)\right] = \prod_{i=1}^{k} E[f_i(\tilde{N}_i)]$$

Hence $N_{C_1}, \dots, N_{C_k}$ are independent N_{C_i} PPP(μ_{C_i}). ∎

7. SIMPLE POISSON PROCESSES.

Def: A measure $\eta \in \mathcal{N}$ is said to be <u>simple</u> if

$$\forall x \in E \quad \eta(\{x\}) \leq 1.$$

Prop: The set

$$\{\eta \in \mathcal{N}_{<\infty} : \eta \text{ is simple}\}$$

is measurable in $\mathcal{N}$.

Proof: Recall the definition of $\tau : \mathcal{N}_{<\infty} \longrightarrow \mathbb{N}$ and $X_i : \mathcal{N}_{<\infty} \longrightarrow E$

in such a way that $\forall \eta \in \mathcal{N}_{<\infty}$ $\eta = \sum_{i=1}^{\tau(\eta)} \delta_{X_i(\eta)}$.

Therefore $\{\eta \in \mathcal{N}_{<\infty} : \eta \text{ is simple}\}$

$$= \{\eta \in \mathcal{N}_{<\infty} ; \forall i < j \leq \tau(\eta) \quad X_i(\eta) \neq X_j(\eta)\}$$

is measurable ∎

Prop: Assume $(E, \mathcal{E})$ is a Polish space and ν is a diffuse ($\nu(\{x\}) = 0 \ \forall x \in E$) and σ-finite measure.

Then every PPP(ν) N is simple a.s.

↳ $\exists \tau, X_i$ r.v. s.t. a.s. $\left(N = \sum_{i=1}^{\tau} \delta_{X_i} \text{ and } X_i \text{ disjoint}\right)$.

Proof: Let $B_i \uparrow E$ s.t. $\forall i \ \mu(B_i) < \infty$.

Consider $\mu_i := \mu(\cdot \cap B_i)$. (NB: μ_i is diffuse)

Let $\tau \sim \text{Poiss}(\mu(B_i))$

$$X_1, X_2, \dots \ \text{iid} \quad X_j \sim \frac{\mu_i[\cdot]}{\mu[B_i]}.$$

As in Section 3, the point process $\tilde{N}_i$ defined by $\tilde{N}_i = \sum_{j=1}^{\tau} \delta_{X_j}$ is a $PPP(\mu_i)$.

$$\mathbb{P}[\tilde{N}_i \text{ is not simple}] \leq \mathbb{P}[\exists j \neq k : X_j = X_k]$$

$$\leq \sum_{j \neq k} \underbrace{\mathbb{P}[X_j = X_k]}$$

$$\underset{X_j, X_k \text{ indep.}}{=} \int_E \underbrace{\mathbb{P}[X_j = x]}_{=0} \frac{\mu_i(dx)}{\mu(B_i)}$$

$$= 0$$

Now, let N be any $PPP(\mu)$ on E.

By restriction $N_{B_i} = N(\cdot \cap B_i)$ is a $PPP(\mu_i)$

By uniqueness $(\mathbb{P}_{N_{B_i}} = \mathbb{P}_{\tilde{N}_i})$

$$\mathbb{P}[N_{B_i} \text{ is simple}] = \mathbb{P}[\tilde{N}_i \text{ is simple}] = 1$$

Hence $\mathbb{P}[\bigcap_i N_{B_i} \text{ is simple}] = 1$, which concludes that N is simple a.s.

8 MARKING.

$(F, \mathcal{F}, \nu)$ probability space "space of marks"

Def: Let $N = \sum_{i=1}^{\tau} \delta_{x_i}$ be a PPP(μ)

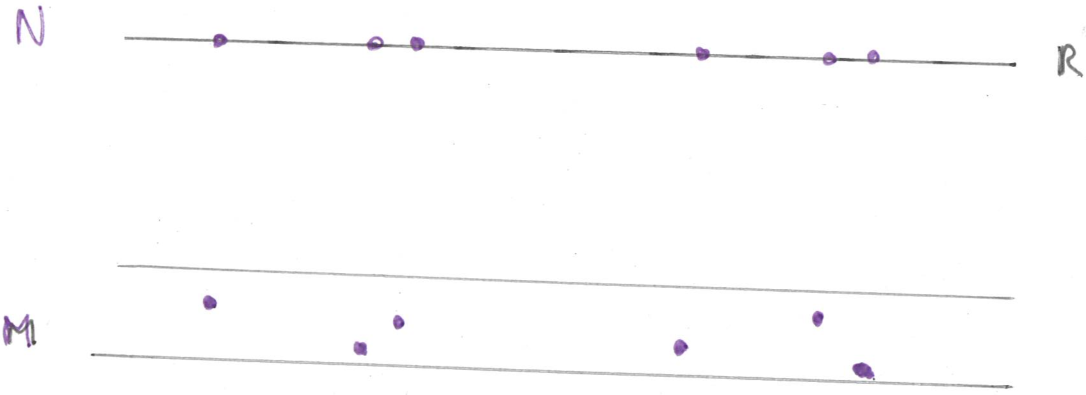
(μ σ-finite measure on $(E, \mathcal{E})$)

$(Y_i)_{i \geq 1}$ iid r.v. with law ν, indep. of N. "ν marking"

The marked point process is the pp on $E \times F$ defined by

$$M = \sum_{i=1}^{\tau} \delta_{(x_i, Y_i)}.$$

Ex1 N PPP (Leb) on $\mathbb{R}$ $\qquad \nu = \mathcal{U}[0,1]$.



Rk: There exist more general markings (where the law of the marks depends on the location of the points)

$\rightarrow$ "probability kernel" see [LAST-PENROSE).

Applications • $X_1, X_2, \dots$ position of trees, $Y_1, Y_2, \dots$ sizes of the trees.

• $X_1, X_2, \dots$ position of cars on the highway, $Y_1, Y_2, \dots$ speeds of the cars.

• Stochastic geometry → see exercises (Boolean percolation).

Thm: $\boxed{\text{The marked process } M \text{ is PPP}(\mu \otimes \nu) \text{ on } E \times F.}$

Proof: M is a p.p. For every $B \subset E$ measurable,

$$M(B) = \sum_{i=1}^{\tau} \underbrace{\mathbb{1}_{(X_i, Y_i) \in B}}_{\text{measurable}} .$$

Let $u : E \times F \longrightarrow \mathbb{R}_+$ measurable.

$$L_M(u) = \sum_{m \in \mathbb{N} \cup \{+\infty\}} \underbrace{E\left[\mathbb{1}_{\tau = m} \exp\left(- \sum_{k=1}^{m} u(X_k, Y_k) \right) \right]}_{=: f(m)}$$

For $m < \infty$, we have

$$f(m) \underset{\text{indep.}}{=} \int_F \cdots \int_F E\left[\mathbb{1}_{\tau = m} \exp\left(- \sum_{k=1}^{m} u(X_k, y_k) \right) \right] \nu(dy_1) \dots \nu(dy_k)$$

$$\overset{\text{Fubini}}{=} E\left[\mathbb{1}_{\tau = m} \prod_{k=1}^{m} \underbrace{\left(\int_F e^{-u(X_k, y_k)} \nu(dy_k) \right)}_{= e^{-v(X_k)}} \right]$$

where $v(x) := -\log\left(\int_F e^{-u(x,y)} \nu(dy) \right) \geq 0$

Hence $\forall m < \infty$ $f(m) = E\left[\mathbb{1}_{\zeta = m} \exp\left(-\sum_{k=1}^{m} v(X_k)\right)\right]$

Equivalently and using convergence monotone, the equality above also holds for $m = +\infty$.

Therefore

$$L_M(u) = \sum_{m \in \mathbb{N} \cup \{\infty\}} E\left[\mathbb{1}_{\zeta = m} \exp\left(-\sum_{k=1}^{m} v(X_k)\right)\right]$$

$$= E\left[\exp\left(-\sum_{k=1}^{\zeta} v(X_k)\right)\right]$$

$$\overset{v \text{ measurable} \geq 0}{=} L_N(v)$$

$$= \exp\left(-\int 1 - e^{-v(x)} \mu(dx)\right)$$

$$= \exp\left(-\int_E \left[\int_F \upsilon(dy) - \int_F e^{-u(x,y)} \upsilon(dy)\right] \mu(dx)\right)$$

$$= \exp\left(-\int_{E \times F} \left(1 - e^{-u(x,y)}\right) \upsilon(dy) \mu(dx)\right).$$

Hence M PPP $(\mu \otimes \upsilon)$. ∎