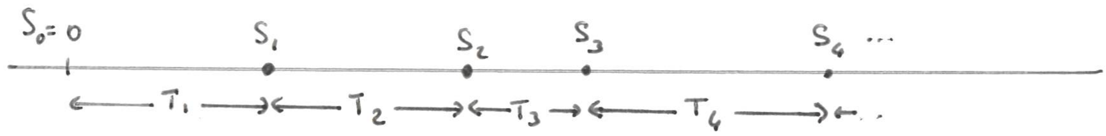
CHAPTER 5.

RENEWAL PROCESSES.

Framework:

- $T_1, T_2, \dots$, iid random variables on $\mathbb{R}_+$ "inter arrival times"
  s.t. $P[T_i = 0] < 1$. $\mu = E[T_1]$.
- $F(t) = P[T_1 \leq t]$ distribution function of T_1.
- $S_n = \sum_{i=1}^{n} T_i \quad (S_0 = 0)$ "renewal times"



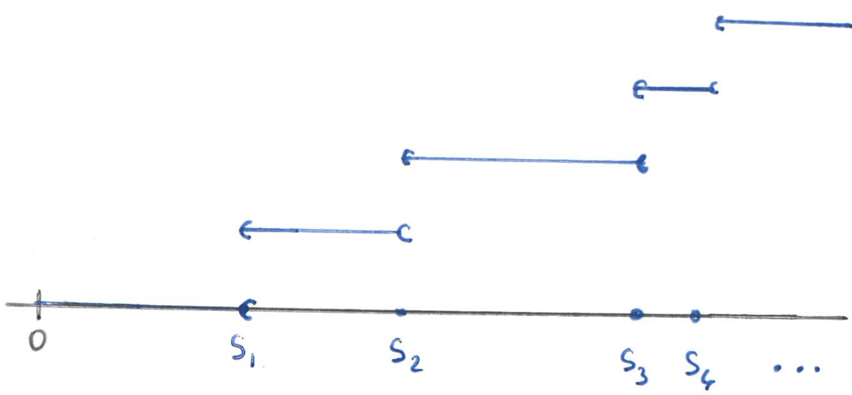
I DEFINITION AND FIRST PROPERTIES

Def: The continuous-time stochastic process $(N_t)_{t \geq 0}$ defined by

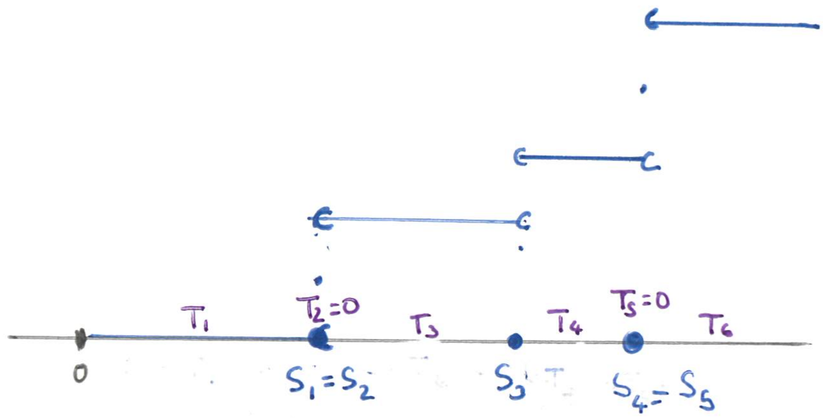
$$\forall t \geq 0 \qquad N_t = \sum_{k=1}^{\infty} \mathbb{1}_{S_k \leq t}$$

is called renewal process with arrival distribution F ($rp(F)$)

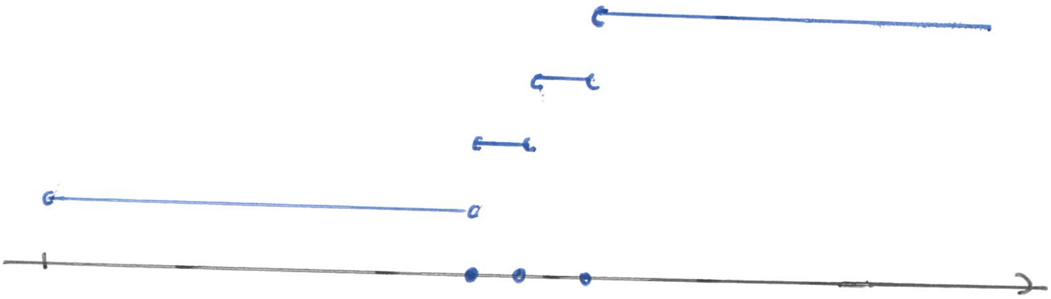
Ex 1 If $T_1 \sim Exp(\lambda)$ then $(N_t)_{t \geq 0}$ is a Poisson process of intensity λ.



Ex2 $\quad T_i = X_i \, T_i'$ where X_i iid Bernoulli$\left(\frac{1}{2}\right)$
$\qquad\qquad\qquad\qquad T_i'$ iid Exp(λ) $\Big)$ independent.

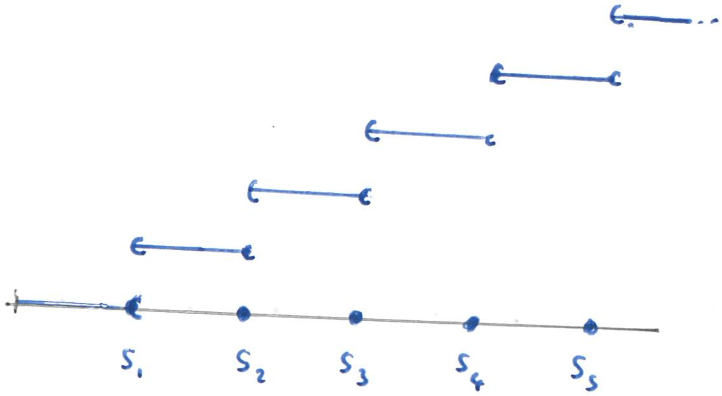


Ex3 $\quad P[T_i \geq t] = \dfrac{1}{\sqrt{1+t}} \mathbb{1}_{t \geq 0} \rightarrow$ "Fat tail" $(p = +\infty)$

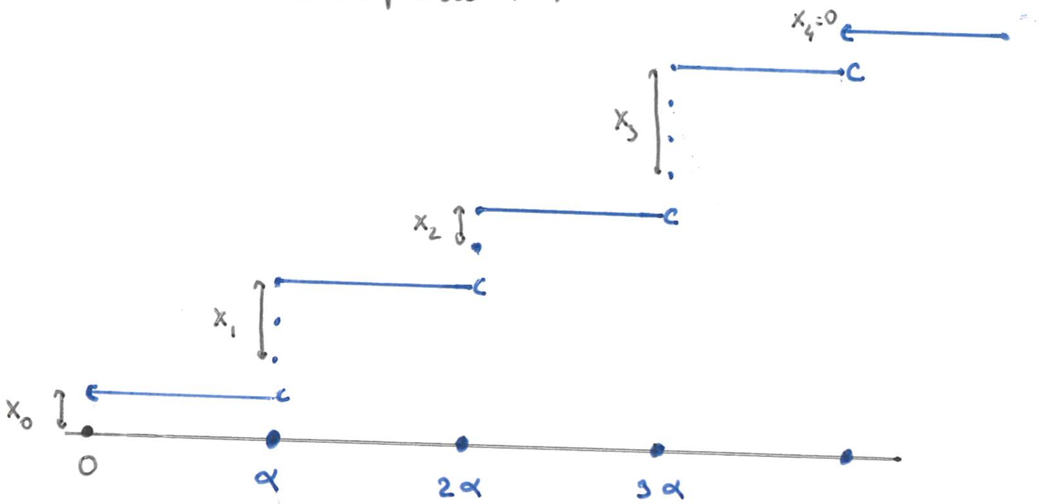


"big jumps"

Ex4 $\quad T_i = 1$ a.s. $\quad$ "deterministic case"



Ex 5. $T_1 = \begin{cases} \alpha & \text{with proba. } \beta \\ 0 & \text{with proba } 1-\beta \end{cases}$, where $\alpha > 0$ $0 < \beta < 1$.



$$\rightarrow N_t \overset{\text{law}}{=} X_0 + \sum_{i=1}^{\lfloor t/\alpha \rfloor} (1 + X_i) \quad \text{where } (X_i) \text{ iid } \mathrm{Geom}(\beta).$$

<u>Prop</u>: $(N_t)_{t \geq 0}$ is a counting process* with jump times $S_1, S_2, \dots$, and $\lim_{t \to \infty} N_t = +\infty$ a.s.

<u>Proof</u>. Since $P[T_1 > 0] > 0$, there exists $\alpha > 0$ s.t $P[T_1 \geq \alpha] > 0$. $\left(\text{Indeed } P[T_1 > 0] = P\Big[\bigcup_{\substack{\alpha > 0 \\ \text{rational}}} \{T_1 \geq \alpha\}\Big] \leq \sum_{\substack{\alpha > 0 \\ \text{rational}}} P[T_1 \geq \alpha] \right)$.

We have

$$\sum_i P[T_i \geq \alpha] = +\infty.$$

Therefore, by Borel-Cantelli lemma, $P[A] = 1$, where

$$A = \{\omega : T_i(\omega) \geq \alpha \text{ for infinitely many } i\text{'s}\}$$

* without the condition that $N_0 = 0$ a.s.

For every $\omega \in A$, $S_n(\omega) \xrightarrow[n\to\infty]{} \infty$, and therefore

$$t \longmapsto N_t(\omega) = \sum_{n \geq 1} \mathbb{1}_{S_n(\omega) \leq t}$$

is a non decreasing function with values in $\mathbb{N}$.

The fact that $N_t \to \infty$ follows from the fact that $\forall i$ $T_i < \infty$ a.s. ■

$$\boxed{\text{Prop: There exists } c > 0 \text{ s.t. } \quad \forall t \geq 0 \quad E[e^{cN_t}] \leq e^{(1+t)/c}}$$

Rk: In particular, for every $t \geq 1$, we have

$$E[e^{c\frac{N_t}{t}}] \underset{\substack{\uparrow \\ \text{Jensen} \\ (x \mapsto x^{1/t} \text{ is concave})}}{\leq} E[e^{cN_t}]^{1/t} \leq e^{2/c}$$

and

$$E\left[\left(\frac{N_t}{t}\right)^k\right] \leq \underbrace{\frac{k!}{c^k} e^{2/c}}_{\text{"uniform bound in } t\text{"}}. \quad \text{"finite moments"}$$

Proof: As in the previous proof, we can pick $\alpha > 0$ s.t. $P[T_1 \geq \alpha] > 0$.

For every $i \geq 1$, define

$$T_i' = \alpha \, \mathbb{1}_{T_i \geq \alpha} .$$

We have $T_i' \leq T_i$ a.s. and (T_i') are iid random variable with

$$T_i' = \begin{cases} \alpha & \text{with probability } \beta \\ 0 & \text{———— } 1-\beta \end{cases}$$

where $\beta = P[T_1 \geq \alpha] > 0$.

Define $S_n' = \sum_{i=1}^n T_i'$ and the renewal process

$$N_t' = \sum_{n \geq 1} \mathbb{1}_{S_n' \leq t} .$$

As in example 5, we have

$$N_t' \overset{law}{=} X_0 + \sum_{i=1}^{\lfloor t/\alpha \rfloor} (1 + X_i)$$

Therefore, for $c > 0$ s.t. $(1-\beta)e^c < 1$, we have

$$\forall t \geq 0 \quad E[e^{cN_t'}] = e^{c\lfloor t/\alpha \rfloor} \times \prod_{i=0}^{\lfloor t/\alpha \rfloor} E[e^{cX_i}]$$

$$\leq e^{ct/\alpha} \times \left(\frac{\beta}{1-(1-\beta)e^c} \right)^{1+t/\alpha}$$

$$\leq \Bigg[\underbrace{\left(\frac{e^c}{1-(1-\beta)e^c} \right)^{1/\alpha}}_{\leq e^{+1/c}} \Bigg]^{1+t}$$

for c small enough. (independent of t)

Theorem [Law of large numbers]

Write $\mu = E[T_1]$. Then

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \quad a.s.$$

Rk: If $\mu = +\infty$, then $\lim_{t\to\infty} \frac{N_t}{t} = 0$ a.s.

Proof: Case 1 : $\mu < \infty$

By the strong law of large number, we have

$$\lim_{n\to\infty} \frac{S_{n+1}}{n+1} = \lim_{n\to\infty} \frac{S_n}{n+1} = \mu \quad a.s$$

Notice that for every t

$$S_{N_t} \leqslant t < S_{N_t+1}.$$

Therefore,

$$\underbrace{\frac{S_{N_t}}{1+N_t}}_{\xrightarrow[a.s.]{} \mu} \leqslant \frac{t}{1+N_t} < \underbrace{\frac{S_{N_t+1}}{N_t+1}}_{\xrightarrow[a.s]{} \mu}.$$

Therefore $\lim_{t\to\infty} \frac{1+N_t}{t} = \frac{1}{\mu}$ a.s.,

which implies that $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu}$ a.s.

Case 2 : $\mu = +\infty$

Define $T_i^{(k)} = \min(k, T_i)$, $k \geq 1$. This way we have $T_i^{(k)} \leq T_i$ a.s. and $T_i^{(k)} \uparrow T_i$ as $k \to \infty$ a.s.
Consider the renewal process $N_t^{(k)}$ associated to the times $(T_i^{(k)})_{i \geq 1}$. Since $E[T_i^{(k)}] \leq k < \infty$, we can apply case 1 to obtain that

$$\forall k \qquad \lim \frac{N_t^{(k)}}{t} = \frac{1}{E[T_1^{(k)}]} \quad \text{a.s.}$$

Since $T_i^{(k)} \leq T_i$ a.s, we have $N_t^{(k)} \geq N_t$ a.s

Hence

$$\forall k \qquad \frac{1}{E[T_1^{(k)}]} \geq \limsup_{t \to \infty} \frac{N_t}{t} \quad \text{a.s.}$$

Now, by monotone convergence, we have

$$\lim_{k \to \infty} E[T_1^{(k)}] = E[T_1] = +\infty .$$

and the two equations above concludes the proof. ■

Thm: [central limit theorem]

Assume that : $E[T_1^2] < \infty$. Write $\mu = E[T_1]$, $\sigma^2 = Var(T_1)$.
Then, assuming $\sigma > 0$, we have.

$$\frac{N_t - t/\mu}{\sigma \sqrt{t/\mu^3}} \xrightarrow[t \to \infty]{\text{law}} \mathcal{N}(0,1) .$$

Application of the law of large numbers: renewal reward process.

Let $(D_i)_{i \geq 1}$ iid r.v. with $D_i \geq 0$, $E[D_i] < \infty$.

Define for every t

$$R_t = \sum_{i \geq 1} D_i \, \mathbb{1}_{S_i \leq t}$$

"reward process"

→ D_i is the "reward" at time S_i.

Then, for t large

$$\frac{R_t}{t} = \underbrace{\frac{1}{N_t} \sum_{i=1}^{N_t} D_i}_{\downarrow \text{ LLG} \atop E[D_1]} \times \underbrace{\frac{N_t}{t}}_{\downarrow \atop 1/\mu}$$

Therefore $\frac{R_t}{t} \longrightarrow \frac{E[D_1]}{\mu}$ a.s.

2 RENEWAL FUNCTION.

Def: The renewal function is defined by $\boxed{\forall t \geq 0 \quad m(t) = E[N_t]}$.

Rk: $E[N_t] < \infty$ (it has exponential moment)

Motivation: $m(t) = E[\# \text{ points in the interval } [0,t]]$

"renewal times"

Prop m is right continuous, non decreasing, non negative

Proof: We have $N_{t+s} - N_t \downarrow 0$ as $s \downarrow 0$. Therefore

$m(t+s) - m(t) \xrightarrow[s \downarrow 0]{} 0$ by monotone convergence.

The other properties are obvious ∎

Exercise: draw $m(t)$ for the 5 examples in Section 1.

Rk: The proposition above implies that there exists a unique measure ν on $\mathbb{R}_+$ s.t.

$$\forall t \quad \nu([0,t]) = m(t). \qquad \text{"Lebesgue - Stieltjes"}$$

$\hookrightarrow$ $\nu(B) = E[\#$ "points" in the set $B]$ for B measurable.

Notation "Lebesgues - Stielgies integral"

Let G be a right continuous non decreasing function on $\mathbb{R}^+$.

Write dG the corresponding Lebesgue Stiltjes measure.

and for $h \in L^1(dG)$ or h measurable non-negative

write

$$\int_{\mathbb{R}_+} h \, dG = \int_{\mathbb{R}_+} h(x) \, dG(x)$$

for the corresponding integral.

Def. [a convolution operator]

Let G be a right continuous non decreasing function on $\mathbb{R}_+$

Let $h : \mathbb{R}_+ \to \mathbb{R}$ s.t. $\forall t \geq 0 \int_0^t |h(t-s)| \, dG(s) < \infty$ or h measurable nonnegative. For every $t \geq 0$, define

$$(h * G)(t) = \int_0^t h(t-s) \, dG(s)$$

Rk: If X, Y are two independent r.v. on $\mathbb{R}_+$ with distribution functions F_X, F_Y resp., then

$$F_{X+Y} = F_X * F_Y.$$

(exercise)

Not. $F^{*k} = \underbrace{F * \cdots * F}_{k \text{ times}}$. distribution function of $S_n = T_1 + \cdots + T_n$.

Prop: For every $t \geq 0$

$$m(t) = \sum_{k \geq 1} F^{*k}(t)$$

Proof: $m(t) = E\left[\sum_{k \geq 1} \mathbb{1}_{S_k \leq t}\right] = \sum_{k \geq 1} P[S_k \leq t] = \sum_{k \geq 1} F^{*k}(t)$ ∎

Thm: [Elementary renewal theorem]

We have $\boxed{\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}}$

Proof: We have $\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{\mu}$ a.s. Furthermore, we have seen that $\sup_{t \geq 1} E\left[\left(\frac{N_t}{t}\right)^2\right] < \infty$. Hence $\frac{N_t}{t}$ is uniformly integrable and

$$\lim_{t \to \infty} \frac{m(t)}{t} = \lim_{t \to \infty} E\left[\frac{N_t}{t}\right] = E\left[\lim_{t \to \infty} \frac{N_t}{t}\right] = \frac{1}{\mu}$$

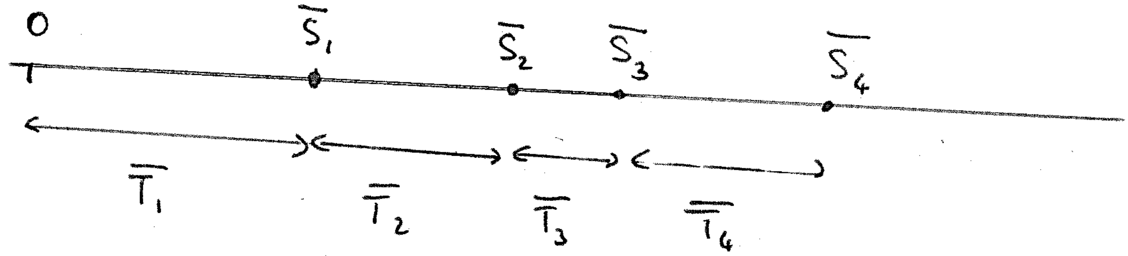
3 RENEWAL WITH DELAY.

↳ in general, the number of arrival in $[a, a+t]$ depends on a ⟶ "no stationary increments"

<u>idea</u>: introduce a delay ⟶ the time $\overline{T}_1$ is chosen with a different law from $\overline{T}_2, \overline{T}_3, \dots$

We consider $(\overline{T}_i)_{i \geq 1}$ independent r.v. on $\mathbb{R}^+$ with

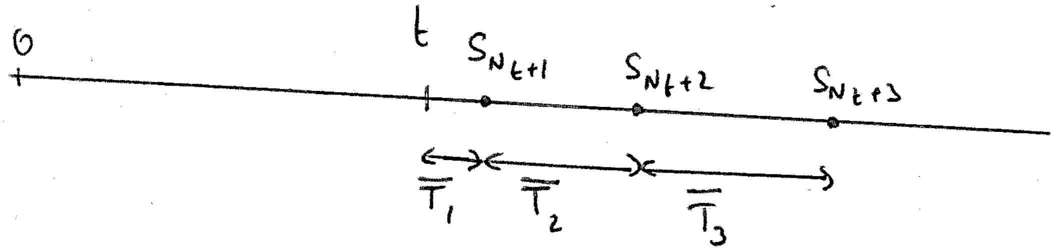
- $\overline{T}_1 \sim dG$
- $\overline{T}_i \sim dF$ for $i \geq 2$



<u>Example</u>: shifted renewal with delay.

$T_1, T_2, \dots$ iid as in the previous sections. Fix $t > 0$

Define then
$$\overline{T}_1 = S_{N_t+1} - t$$
$$\overline{T}_i = S_{N_t+i} - S_{N_t+i-1}, \quad i \geq 2$$



Def: $\bar{S}_i = \bar{T}_1 + \dots + \bar{T}_i \quad , \; i \geq 1$

$$\bar{N}_t = \sum_{i \geq 1} \mathbb{1}_{\bar{S}_i \leq t} \quad , \; t \geq 0$$

$(\bar{N}_t)_{t \geq 0}$ is called a renewal process with distr. function F and delay function G.

Goal compute $\bar{m}(t)$ the renewal function associated to $\bar{N}_t$ and find G, s.t. $\bar{m}$ is linear.

Not: $\bar{m}(t) = E[\bar{N}_t]$

As in the previous section, we can prove the following

Prop:

$$\boxed{\forall t \geq 0 \quad \bar{m}(t) = \sum_{i \geq 0} G * F^{*i}(t)}$$

The Laplace transform behaves "nicely" with the convolution operator, and the Laplace transform of $\bar{m}$ can easily be computed.

Not: $\mathcal{M} = \{ f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ right continuous, nondecreasing} \}$

"$\mathcal{M}$ for nonnegative Measure"

$f \in \mathcal{M} \xleftrightarrow[\text{Lebesgue Stieltjes}]{} df$ nonneg. measure on $\mathbb{R}_+$

Def: Let $f \in \mathcal{M}$. For every $s \geq 0$, define

$$Lf(s) = \int_0^{\infty} e^{-sx}\, df(x).$$

Rk: If $f = F_Y$ distribution function of a non-negative r.v. Y, then $L_f(s) = E[e^{-sY}]$.

Prop: For every $f, g \in \mathcal{M}$, we have

$$L_{f * g} = L_f \times L_g$$

Rk: If X, Y are two indep. r.v. on $\mathbb{R}_+$

$$\hookrightarrow E[e^{-s(X+Y)}] = E[e^{-sX}]\, E[e^{-sY}]$$

Rk: If $f, g \in \mathcal{M}$, $f * g$ is well defined and $f * g \in \mathcal{M}$.

Proof: For every $h \geq 0$ measurable, we have

$$\int h\, d(f*g) = \iint h(x+y)\, df(x)\, dg(y)$$

$\to$ true for $h = \mathbb{1}_{(a,b]}$ because

$$\begin{aligned}(f*g)(b) - (f*g)(a) &= \int f(b-y) - f(a-y)\, dg(y) \\ &= \int \int \mathbb{1}_{x \in (a-y,\, b-y]}\, df(x)\, dg(y) \\ &= \iint \mathbb{1}_{(a,b]}(x+y)\, df(x)\, dg(y)\end{aligned}$$

$\to$ true for general $h \geq 0$ by approximation.

In particular for $h(x) = e^{-sx}$, we have

$$\int e^{-sz}\, d(f * g)(z) = \iint e^{-s(x+y)}\, df(x)\, dg(y)$$

$$\underset{\text{Fub.}}{=} \int e^{-sx}\, df(x) \int e^{-sy}\, dg(y) \quad \blacksquare$$

Corollary:

$$\boxed{L_{\bar{m}} = \frac{L_G}{1 - L_F}}$$

Proof: By monotone convergence, we have

$$L_{\bar{m}} = \sum_{i \geq 0} L_{G * F^{*i}}$$

By induction $L_{G * F^{*i}} = L_G \times L_F^i$.

Hence $\forall t > 0$

$$L_{\bar{m}}(t) = \sum_{i \geq 0} L_G(t) \cdot L_F^i(t) = L_G(t) \cdot \sum_{i \geq 0} L_F^i(t)$$

$$= \frac{L_G(t)}{1 - L_F(t)} \qquad (\text{because } L_F(t) < 1)$$

The equality extends to $t = 0$ since both terms are infinite. $\blacksquare$

Def: Consider the delay function defined by

$$\forall t \geq 0 \quad G_*(t) = \frac{1}{\mu} \int_0^t (1 - F(x))\, dx.$$

Rk: $\int_0^\infty (1-F(x))\, dx = \int_0^\infty P[T_1 > x]\, dx$

$$\overset{\text{Fub}}{=} E\left[\int_0^\infty \mathbb{1}_{T_1 \geq x}\, dx\right] = E[T_1] = \mu.$$

Hence $\lim_{t \to \infty} G_*(t) = 1$ and G_* is indeed a distribution function.

Thm: Assume $\mu < \infty$.
For the renewal process with delay function G_*, we have

$$\bar{m}(t) = \frac{t}{\mu} \quad \text{for every } t \geq 0$$

("the process is stationary")

Lemma: Let $m_1, m_2 \in \mathcal{M}$ and assume that

$$\forall t > 0 \quad L_{m_1}(t) = L_{m_2}(t) < \infty$$

Then $m_1 = m_2$

Proof: admitted.

Proof of the theorem: For $s > 0$

Notice that for every h measurable bounded

$$\int_0^\infty h(x)\, dG_0(x) = \int_0^\infty h(x)(1-F(x))\frac{dx}{\mu}$$

(as usual, start by $h = \mathbb{1}_{[a,b]}$ and conclude by approximation).

In particular, for every $s \geq 0$

$$L_{G_0}(s) = \int_0^\infty e^{-sx}\, dG_0(x) = \int_0^\infty e^{-sx}\underbrace{(1-F(x))}_{=P[\overline{T}_2 > x]}\frac{dx}{\mu}$$

$$= \frac{1}{s\mu}\int_0^\infty s e^{-sx} P[\overline{T}_2 > x]\, dx$$

$$\overset{\text{Fub}}{=} \frac{1}{s\mu} E\left[\int_0^\infty s e^{-sx}\mathbb{1}_{\overline{T}_2 > x}\, dx\right]$$

$$= \frac{1}{s\mu} E\left[\int_0^{\overline{T}_2} s e^{-sx}\, dx\right]$$

$$= \frac{1}{s\mu} E\left[1 - e^{-s\overline{T}_2}\right] = \frac{1 - L_F(s)}{s\mu}$$

Therefore $L_{\bar{m}}(s) = \dfrac{1 - L_F(s)}{s\mu} \times \dfrac{1}{1 - L_F(s)}$

$$= \frac{1}{s\mu} = \frac{1}{\mu} \int_0^\infty e^{-sx}\, dx.$$

$$= L_{\frac{1}{\mu}\mathrm{Id}} .$$

By the lemma, we conclude $\forall t \geq 0 \;\; \bar{m}(t) = \frac{1}{\mu} t$ ■

4 BLACKWELL'S RENEWAL THEOREM

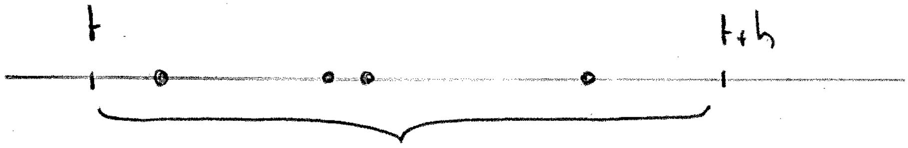
Def: We say that the law of T_1 is non arithmetic if

$$\forall a > 0 \quad P[T_1 \in a\mathbb{Z}] < 1.$$

Thm [Blackwell's renewal theorem]

> Assume that the law of T_1 is non-arithmetic. Then
>
> $$\forall h \geq 0 \qquad \lim_{t\to\infty} m(t+h) - m(t) = \frac{h}{\mu}.$$

Interpretation



$$E[\#\text{points}] = \frac{1}{\mu} \cdot h$$

Sketch of proof for $\mu < \infty$.

Consider $\overline{T}_i$, $i \geq 1$ ind. and ind of $(T_i)_{i \geq 1}$ where $\overline{T}_1 \sim dG_*$ (as defined in Section 3)

$$\overline{T}_i \sim dF, \quad i \geq 2.$$

Then the renewal function associated to these interarrival times is

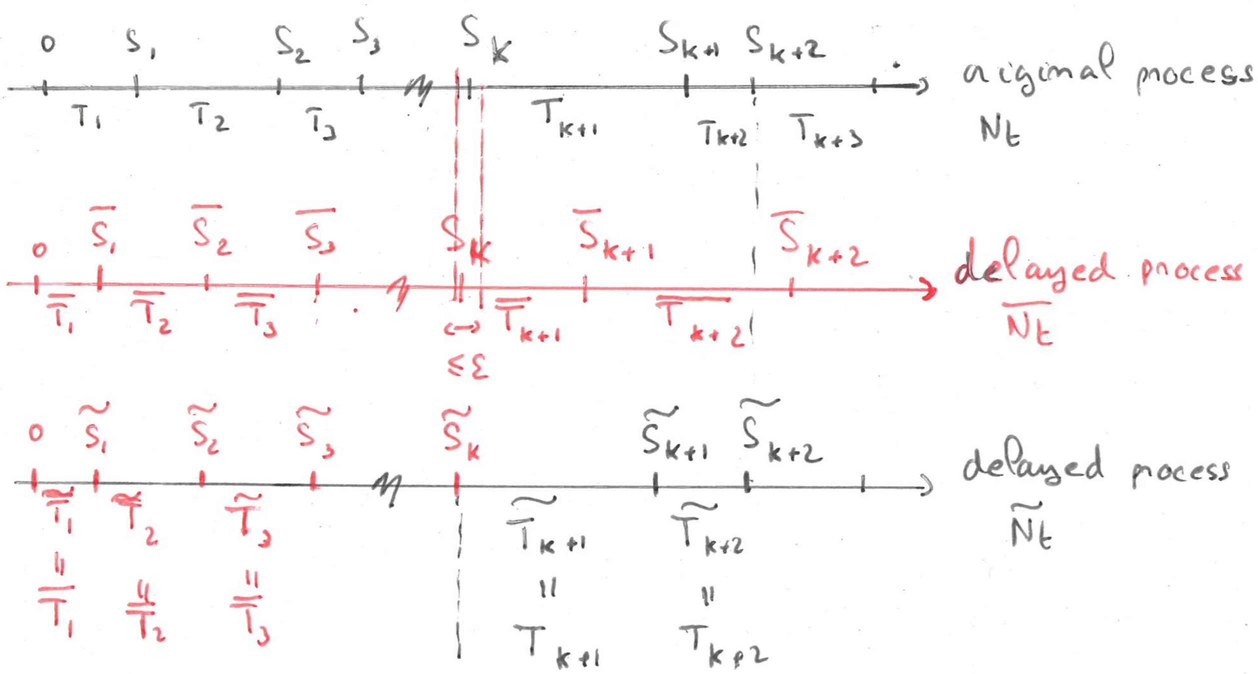
$$\forall t \quad \overline{m}(t) = \frac{1}{\mu} t. \qquad \text{"stationary"}$$

Write $S_k = \sum_{i \leq k} T_i$, $\overline{S}_k := \sum_{i \leq k} \overline{T}_i$.

<u>Claim</u>: Let $\varepsilon > 0$, then a.s. there exists $K > 1$ (random) s.t.

$$|S_K - \overline{S}_K| \leq \varepsilon.$$

(admitted).



Consider

$$\tilde{T}_i = \begin{cases} \bar{T}_i & , i \leq k \\ T_i & , i > K \end{cases}$$

$\hookrightarrow$ the renewal process associated to $(\tilde{T}_i)$ is a delayed process with renewal function

$$\tilde{m}(t) = \frac{t}{\mu} .$$

Furthermore for t large $(t > k)$, we have

$$N_{t+h} - N_t \simeq \tilde{N}_{t+h} - \tilde{N}_t$$

Therefore

$$m(t+h) - m(t) = E[N_{t+h} - N_t]$$

$$\simeq E[\tilde{N}_{t+h} - \tilde{N}_t] = \frac{h}{\mu} \quad \tilde{\blacksquare}$$

5 THE RENEWAL EQUATION

$\boxed{\begin{array}{l}\textbf{Def}: \text{ Let } h : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ measurable, loc. bounded} \\ \text{Let } g : \mathbb{R}_+ \to \mathbb{R} \text{ s.t } \forall t \geq 0 \ \int_0^t |g(t-s)| \, dF(s) < \infty . \\ \text{We say that } g \text{ is a solution of the } (h, F)\text{-renewal-equation (ren. eq.) if} \\ \boxed{\forall t \geq 0 \quad g(t) = h(t) + \int_0^t g(t-s) \, dF(s)} \\ \text{i.e: } g = h + g * F \end{array}}$

First example:

Prop: m is solution of the (F,F) - renewal equation:

$$m = F + m * F$$

Pf. 1:

$$m = \sum_{i\geq 1} F^{*i} = F + \sum_{i\geq 2} F^{*i} * F$$

$$= F + \underbrace{\Big(\sum_{i\geq 2} F^{*(i-1)}\Big)}_{m} * F \quad \blacksquare$$

Pf 2: For $t \geq 0$, we have

$$m(t) = E\Big[\sum_{k\geq 1} \mathbb{1}_{T_1+\dots+T_k \leq t}\Big]$$

$$= P[T_1 \leq t] + \underbrace{E\Big[\sum_{k\geq 2} \mathbb{1}_{T_1+T_2+\dots+T_k \leq t}\Big]}$$

$$\overset{\text{Fub}}{=} \sum_{k\geq 2} E\big[\mathbb{1}_{T_1+T_2+\dots+T_k \leq t}\big]$$

$$\overset{\text{indep}}{=} \sum_{k\geq 2} \int_0^t E\big[\mathbb{1}_{s+T_2+\dots+T_k \leq t}\big]\, dF(s)$$

$$\overset{\text{Fub}}{=} \int_0^t E\Big[\sum_{k\geq 2} \mathbb{1}_{T_2+\dots+T_k \leq t-s}\Big]\, dF(s)$$

$$= \int_0^t m(t-s)\, dF(s) \quad \blacksquare$$

Example 2 : generalization

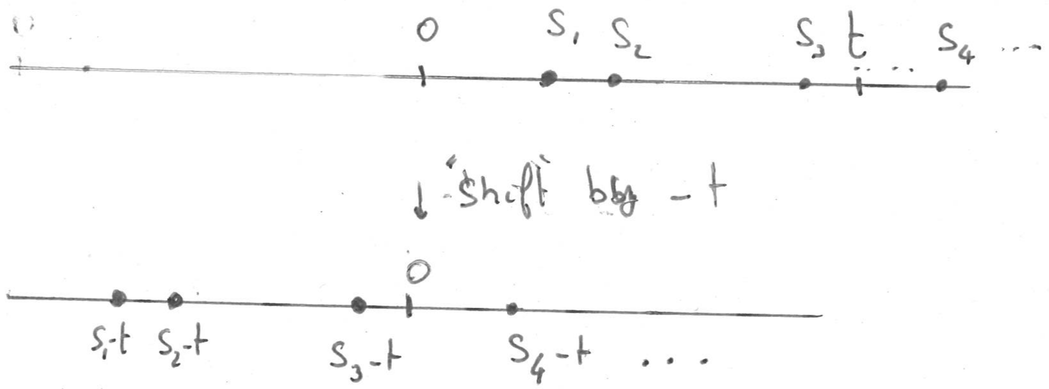
Let $E = \{ (s_i)_{i \geq 1} : s_1 \leq s_2 \leq \dots \, , \, s_i \xrightarrow[i \to \infty]{} \infty \} \subset \underbrace{\mathbb{R}^{\mathbb{N}}}$

equiped with product σ-algebra.

Let $\Phi : E \longrightarrow \mathbb{R}$ measurable s.t.

$$\forall t \geq 0 \, , \, i = 1,2 \qquad E\left[\left| \Phi(S_i - t, S_{i+1} - t, \dots) \right| \right] < \infty$$

Define $\boxed{\varphi(t) = E\left[\Phi(S_1 - t, S_2 - t, \dots) \right] , \; t \geq 0}$



Prop. φ is solution of the (h, F)-ren. eq.,

where

$$\forall t \geq 0 \quad h(t) = E\left[\Phi(S_1 - t, S_2 - t, \dots) \right]$$

$$- E\left[\Phi(S_2 - t, S_3 - t, \dots) \mathbb{1}_{T_1 \leq t} \right]$$

ie : $\boxed{\forall t \geq 0 \quad \varphi(t) = h(t) + \int_0^t \varphi(t-s) \, dF(s)}$

Proof: $E[\,\Phi(S_1-t, S_2-t, \dots)]$

$$= h(t) + E[\Phi(S_2-t, S_3-t, \dots)\,\mathbb{1}_{T_1 \le t}]$$

$$= h(t) + E[\Phi(T_1+T_2-t, T_1+T_2+T_3-t, \dots)\,\mathbb{1}_{T_1\le t}]$$

$$\overset{\text{ind.}}{=} h(t) + \int_0^t \underbrace{E[\Phi(s+T_2-t, s+T_2+T_3-t, \dots)]}_{=\varphi(t-s)}\,dF(s)$$

$$= h(t) + \int_0^t \varphi(t-s)\,dF(s)$$

■

Appli. 1 m is solution of the (F, F)-ren. eq.

$N_t = \Phi(S_1-t, S_2-t, \dots)$ where $\Phi(s_1, s_2, \dots) = \sum_i \mathbb{1}_{s_i \le 0}$

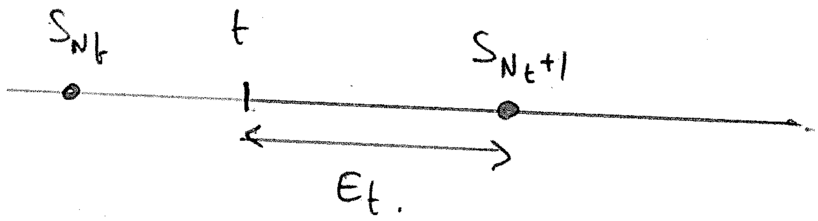
Hence $m(t) = E[N_t]$ is sol. of the (h, F) ren. eq.

with $h(t) = E[\underbrace{\Phi(S_1-t, S_2-t, \dots) - \mathbb{1}_{T_1\le t}\,\Phi(S_2-t, S_3-t, \dots)}_{= \begin{cases} 0 & \text{if } T_1 > t \\ 1 & \text{if } T_1 \le t \end{cases}}]$

ie $h(t) = P[T_1 \le t]$

Appli 2 : Excess time.

For $t \geq 0$, define $E_t = S_{N_t+1} - t$.



For $x \geq 0$, let $e_x(t) = P[E_t \leq x]$., $t \geq 0$

Then for every $t \geq 0$

$$e_x(t) = F(t+x) - F(t) + \int_0^t e_x(t-s)\, dF(s)$$

Proof: $e_x(t) = E\left[\Phi(S_1 - t, S_2 - t, \dots) \right]$

where $\Phi(s_1, s_2, \dots) = \mathbb{1}_{(\min\{s_i : s_i \geq 0\}) \leq x}$.

e_x is sol. of the (h, F)-eq., where

$$h(t) = E\Big[\underbrace{\Phi(S_1 - t, \dots) - \mathbb{1}_{T_1 \leq t}\, \Phi(S_2 - t, \dots)}\Big]$$

$$= \begin{cases} 0 & \text{if } T_1 \leq t \text{ or } T_1 > t+x \\ 1 & \text{if } t < T_1 \leq t+x \end{cases}$$

$$= P[t < T_1 \leq t+x] = F(t+x) - F(t) \qquad \blacksquare$$

5.2 WELL-POSEDNESS OF THE RENEWAL EQUATION.

Thm [existence and uniqueness]

Let $h : \mathbb{R}_+ \to \mathbb{R}$ meas. loc. bounded.

Then there exists a unique $g : \mathbb{R}_+ \to \mathbb{R}$ measurable loc. bounded, solution of

$$g = h + g * F,$$

given by

$$g = h + h * m.$$

Proof: existence.

Let $g = h + h * m$. (g is measurable loc. bounded, because h is). We have

$$h + g * F = h + (h + h * m) * F$$

$$= h + h * \underbrace{(F + m * F)}_{= m} = g.$$

uniqueness

Let g_1, g_2 be two solutions of the (h, F) ren. eq.

Then $g_1 - g_2 = (g_1 - g_2) * F \underset{\substack{\uparrow \\ \text{induction}}}{=} (g_1 - g_2) * F^{*n}$

For every $t \geq 0$

$$|g_1(t) - g_2(t)| = \left| \int_0^t (g_1 - g_2)(t-s) \, dF^{*n}(s) \right|$$

$$\leq \sup_{[0,t)} |g_1 - g_2| \underbrace{\underbrace{\int_0^t dF^{*n}(s)}_{P[T_1 + \dots + T_n \leq t]}}_{\xrightarrow[n \to \infty]{} 0}$$

Hence $g_1 = g_2$ ∎

5.3 ASYMPTOTIC BEHAVIOR

In this section we assume that the law of T_1 is non-arithmetic.

Goal: Let g sol. of (h,F)-ren. eq., what is the asymptotic behavior of $g(t)$ for $t \to \infty$?

Case 1: $h = \mathbb{1}_{[a,b]}$ $0 \leq a < b$. , g sol of (h,F) ren. eq.

$$g(t) = h(t) + \int_0^t h(t-s) \, dm(s)$$

$$= h(t) + \int_{t-b}^{t-a} h(s) \, dm(s) \qquad \text{for } t \text{ large}$$

$$= \underbrace{h(t)}_{\xrightarrow[t \to \infty]{} 0} + \underbrace{m(t-a) - m(t-b)}_{\xrightarrow[t \to \infty]{} \frac{b-a}{\mu}} \qquad \text{for } t \text{ large}$$

(Blackwell)

Hence $\lim_{t\to\infty} g(t) = \frac{1}{\mu}\int_0^\infty h(s)\,ds$ (if $h = \mathbb{1}_{[a,b]}$)

↳ "How does it generalize?".

Def: $h : \mathbb{R}_+ \to \mathbb{R}_+$ measurable is called directly Riemann-integrable (dRi) if

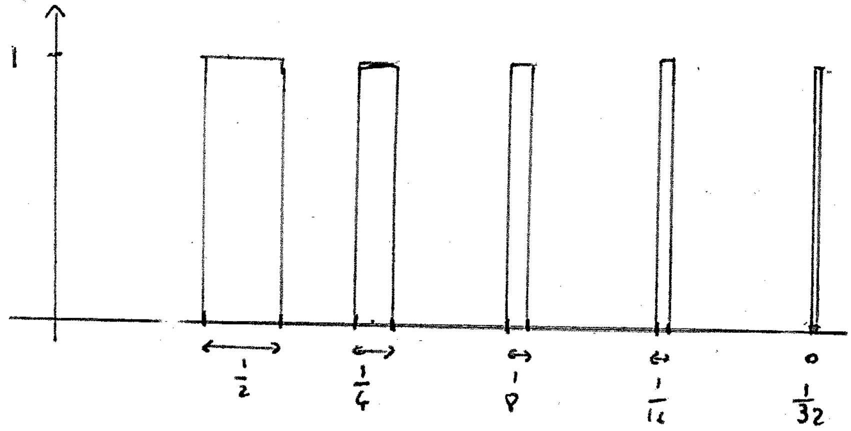
$$\forall \Delta > 0 \quad \sum_{k=0}^{\infty} \sup_{[k\Delta, k\Delta+\Delta]} h < \infty$$

and

$$\lim_{\Delta\to 0} \Delta \sum_{k=0}^{\infty} \sup_{[k\Delta, k\Delta+\Delta]} h = \lim_{\Delta\to 0} \Delta \sum_{k=0}^{\infty} \inf_{[k\Delta, k\Delta+\Delta)} h .$$

$h : \mathbb{R}_+ \to \mathbb{R}$ is dRi if $h_+ = \max(h, 0)$ and $h_- = \max(-h, 0)$ are d.R.i.

Rk: $h = \sum \mathbb{1}_{[k, k+2^{-k}]}$ is not dRi (but is integrable.)



Prop. [sufficient condition]

Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ measurable.

Assume that

- h is continuous at a.e $t \in \mathbb{R}$
- there exists H non increasing s.t. $\int_0^\infty H < \infty$

and
$$0 \leq |h| \leq H .$$

Then h is dRi.

<u>Rk:</u> In particular if h is bounded continuous at a.e. $t \in \mathbb{R}$ and vanishes outside a compact, then h is dRi

<u>Proof</u>: see lecture notes [Sniteman 17]

<u>Thm:</u> [Smith's key renewal thm]

Let h dRi, F non arithmetic. Then $g = h + h * m$ satisfies

$$\lim_{t \to \infty} g(t) = \frac{1}{\mu} \int_0^\infty h(u)\, du .$$

Proof: Since h is dRi we have

$$\sum_k \sup_{[k,k+1)} h < \infty$$

Hence $h(t) \xrightarrow[t\to\infty]{} 0$.

Therefore it suffices to prove

$$\lim_{t\to\infty} \int_0^t h(t-s)\, dm(s) = \frac{1}{\gamma} \int_0^t h(u)\, du.$$

Let $\Delta > 0$ s.t. $F(\Delta) < 1$

- Assume $h = \sum_{k\geq 0} c_k \mathbb{1}_{[k\Delta, k\Delta+\Delta)}$ $\quad c_k \geq 0 \quad \sum c_k < \infty$

By monotone convergence

$$\int h(t-s)\, dm(s) = \sum_{k\geq 0} \underbrace{c_k \left[m(t-k\Delta) - m(t-k\Delta-\Delta)\right]}_{h_k(t)}$$

Observe that for every $u \geq \Delta$

$$1 \geq F(u) = m(u) - \int_0^u F(u-s)\, dm(s) \quad \left(\begin{array}{l}\text{because}\\ m \text{ sol. of } (F,F)\text{-eq.}\end{array}\right)$$

$$= \int_0^u \big(1 - F(u-s)\big)\, dm(s)$$

$$\geq \int_{u-\Delta}^u \underbrace{\big(1 - F(u-s)\big)}_{\geq 1-F(\Delta)}\, dm(s) \geq \big(1 - F(\Delta)\big)\big(m(u) - m(u-\Delta)\big)$$

Hence for every t and every k

$$h_k(t) \leqslant \frac{c_k}{1-F(\Delta)}.$$

(distinguish between $t-k\Delta \geqslant \Delta$ and $t-k\Delta < \Delta$ and use that m is non decreasing, vanishing on $(-\infty, 0)$)

By dominated convergence

$$\lim_{t\to\infty} \sum_k h_k(t) = \sum_k \underbrace{\lim_{t\to\infty} h_k(t)}_{\substack{= c_k \times \frac{\Delta}{\mu} \\ \uparrow \\ \text{(Blackwell)}}}$$

Hence $\lim\limits_{t\to\infty} \int h(t-s)\, dm(s) = \sum\limits_{k=0}^{\infty} c_k \frac{\Delta}{\mu} = \frac{1}{\mu}\int_0^\infty h(u)\, du$

- Assume $h \geqslant 0$ dRi. Le $\Delta > 0$ s.t. $F(\Delta) < 1$.

Write $\underline{h}_\Delta = \sum\limits_{k\geqslant 0} \left(\inf\limits_{[k\Delta, k\Delta+\Delta)} h \right) \mathbb{1}_{[k\Delta, k\Delta+\Delta)}$

$$\overline{h}_\Delta = \sum_{k\geqslant 0} \left(\sup_{[k\Delta, k\Delta+\Delta)} h \right) \mathbb{1}_{[k\Delta, k\Delta+\Delta)}.$$

We have for every t

$$\int_0^t h(t-s)\,dm(s) \leqslant \underbrace{\int_0^t \overline{h}_\Delta(t-s)\,dm(s)}_{\xrightarrow[t\to\infty]{} \frac{1}{\mu}\int \overline{h}_\Delta(u)\,du}$$

(by the previous step)

Hence

$$\limsup_{t\to\infty} \int_0^t h(t-s)\,dm(s) \leqslant \frac{1}{\mu}\int_{\mathbb{R}} \overline{h}_\Delta(u)\,du$$

Since $\left|\int_{\mathbb{R}} \overline{h}_\Delta(u)\,du - \int h(u)\,du\right| \leqslant \sum_{k\geq 0} \Delta\left(\overline{h}_\Delta(k\Delta) - \underline{h}_\Delta(k\Delta)\right)$

$$\xrightarrow[\Delta\to 0]{} 0 \qquad \text{(because } h \text{ d.Ri)}$$

we can let Δ tend to 0 in the equation above to obtain

$$\limsup_{t\to\infty} \int_0^t h(t-s)\,dm(s) \leqslant \frac{1}{\mu}\int_{\mathbb{R}} h(u)\,du$$

Equivalently,

$$\liminf_{t\to\infty} \int_0^t h(t-s)\,dm(s) \geqslant \frac{1}{\mu}\int_{\mathbb{R}} h(u)\,du$$ ∎

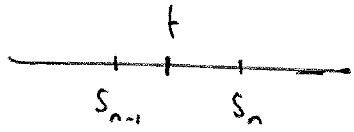
Proof of invariance of delayed R.P.

Pick $\tilde{T}_1$ with dist fct $dG = \overbrace{\frac{t}{\mu}}^{\text{“size biased time”}} dF$ $\quad \left(\mathbb{E} f(\tilde{T}_1) = \frac{1}{\mu} \mathbb{E}[f(T_1) \cdot T_1] \right)$

Let $U \sim U[0,1)$

Delayed process $U\tilde{T}_1, T_2, T_3, \ldots \longrightarrow \tilde{N}_t = \sum_n \mathbb{1}_{U\tilde{T}_1 + T_2 + \cdots + T_n \leq t}$

$$\tilde{m}(t) = \frac{1}{\mu} \sum_n \mathbb{E}\left[T_1 \mathbb{1}_{UT_1 + T_2 + \cdots + T_n \leq t}\right]$$



$$= \frac{1}{\mu} \sum_n \underbrace{\mathbb{E}\left[T_n \mathbb{1}_{T_1 + \cdots + T_{n-1} + UT_n \leq t}\right]}$$

$(T_1, \ldots, T_n) \overset{law}{=} (T_n, T_2, \ldots, T_{n-1}, T_1)$

$$= \mathbb{E}\Big[T_n \underbrace{\mathbb{P}[S_{n-1} + UT_n \leq t \mid T_1, \ldots, T_n]}_{\mathbb{1}_{S_n \leq t} + \mathbb{1}_{S_{n-1} \leq t \leq S_n} \frac{t - S_{n-1}}{T_n}}\Big]$$

$$= \frac{1}{\mu} \mathbb{E}[S_{N_t}] + \frac{1}{\mu} \mathbb{E}[t - S_{N_t}]$$

$$= \frac{t}{\mu} \quad \blacksquare$$