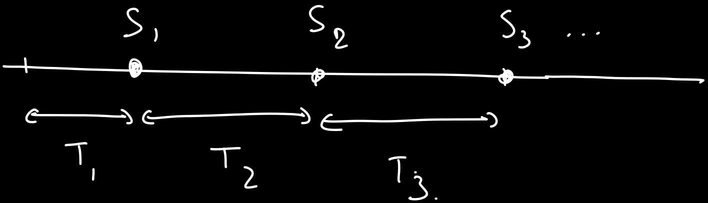
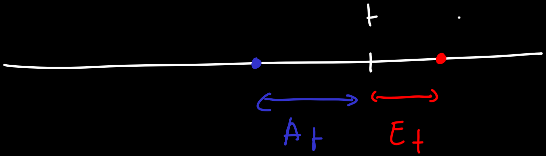


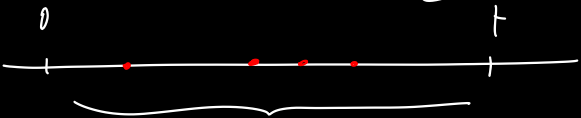
T_i iid → $F(t) = \mathbb{P}(T_1 \leq t)$

$\mu = E(T_1)$

$S_i = T_1 + \dots + T_i$



$$N_t = \sum_n \mathbb{1}_{S_n \le t}$$



$N_t = \#$ renewal.

2. RENEWAL FUNCTION.

Def 1 The renewal function is defined by $\boxed{m(t) = \mathbb{E}[N_t]}$ $(t \ge 0)$

$m(t) = E[\text{\# renewal times in } [0,t]]$.

Prop: $m(t)$ is
- non decreasing
- non negative
- right continuous.

Pf: right. cont. Since $(N_t)_{t\geq 0}$ is right-cont

$$\lim_{\substack{s\to 0\\ >}} N_{t+s} - N_t = 0$$

Therefore

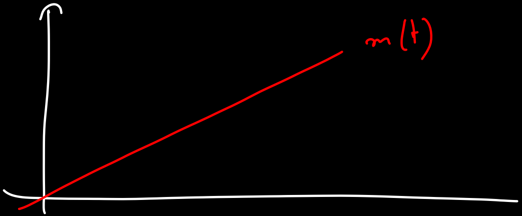
$$\overset{= E[N_{t+s} - N_t]}{\overline{m(t+s) - m(t)}} \xrightarrow[\substack{s\to 0\\ >}]{} 0$$

by monotone convergence.

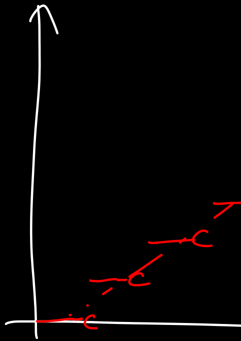
Examples:

. $T_1 \sim \mathrm{Exp}(\lambda)$ $\left(\overset{N_t}{Pp(\lambda)} \right)$

$$m(t) = \mathbb{E}[\underset{\uparrow}{N_t}) = \lambda t = \frac{t}{\mu}$$

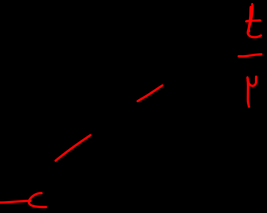


. $T_1 = 1$. a



Exercise: try

. s .



o the distributions

Thm: (Elementary renewal

$$\boxed{\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}}$$

Proof: $\frac{N_t}{t} = \frac{1}{\mu}$ a.s.

Furthermore, we have see

$$\exists c>0 \; \forall t \; E\left[e^{cN_t}\right] \leq$$

$$\boxed{t \geq 1} \qquad E\Big[\underbrace{e^{c\frac{N_t}{t}}}_{\sum \frac{1}{k!} c^k \left(\frac{N_t}{t}\right)^k}\Big] \leq$$

Jens

thm]

that

$e^{\frac{1+t}{c}}$

$\mathbb{E}[e^{cN_t})^{\frac{1}{t}} \leq e^{\frac{2}{c}}$

en

- $\underline{\forall t \geq 1}$

$$\underbrace{\mathbb{E}\left[\left(\frac{N_t}{t}\right)^2\right] \leq \frac{2!}{c^2} \, e}$$

$$\sup_{t \geq 1} : \mathbb{E}\left[\left(\frac{N_t}{t}\right)^2\right]$$

. This implies that

$\frac{N_t}{t}$ is uniformly

Hence $\mathbb{E}\left(\frac{N_t}{t}\right) \xrightarrow[t \to \infty]{}$

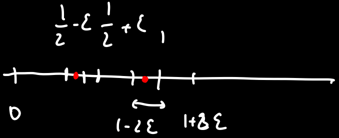
$^2/_c$

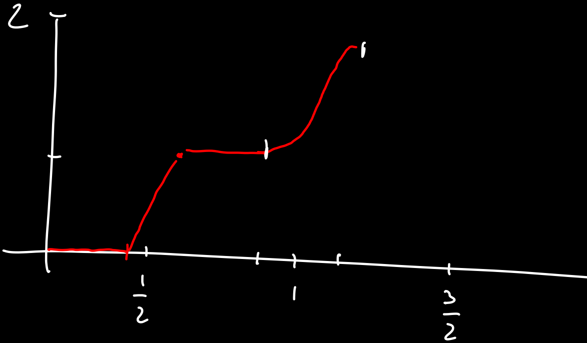
$< \infty$

integrable

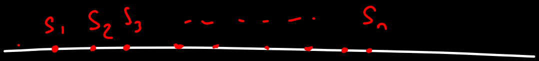
$\frac{1}{\mu}$ ∎

- $\underbrace{T_1 \sim \mathcal{U}\left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right]}$

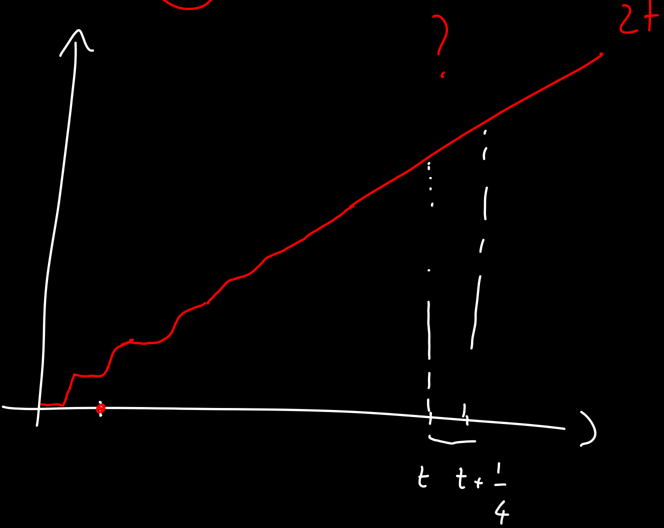




$\varepsilon > 0$



$$S_n = \underbrace{\frac{n}{2}} + \underbrace{\sqrt{\varepsilon n} \cdot X}$$



3. BLACKWELL'S RENEWAL THEOREM

Def: We say that the law of T_1 is arithmetic if

$$\exists\, a > 0 : \mathbb{P}[T_1 \in a\mathbb{Z}] = 1$$

It is non-arithmetic if

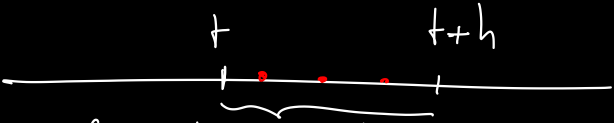
$$\forall a > 0 \quad \mathbb{P}[T_1 \in a\mathbb{Z}] < 1$$

Thm: [Blackwell].

Assume that the law of T_1 is non-arithmetic. Then

$$\forall h>0 \quad \lim_{t\to\infty} m(t+h) - m(t) = \frac{h}{\mu}$$

Interpretation:



$E(\#\text{parts}) = h/\mu$

Rk:

(u kr

$$\frac{m(t)}{t} \simeq \frac{m(\lfloor t \rfloor)}{\lfloor t \rfloor}.$$

$$= \frac{1}{\lfloor t \rfloor} \sum_{k=1}^{\lfloor t \rfloor} \underbrace{m(k) - m(k-1)}_{\overset{\shortparallel}{u_k}}$$

$$\to \frac{1}{\mu} \Big)$$

$$\xrightarrow{\quad} \frac{1}{\mu}$$

$\uparrow$ by Cesaro.

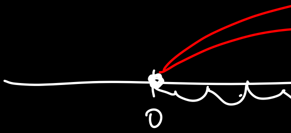
"Blackwell is stronger than elementary renewal"

Rk: A version

distributi

$a_0 := \max\{ a$

$\boxed{?}$ connectic

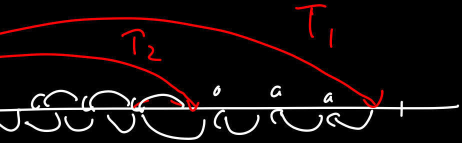


exist for arithmetic

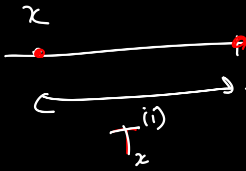
ns (see exercises).

: $\mathbb{P}[\,T_1 \in a\mathbb{Z}\,] = 1\,)$

with period?



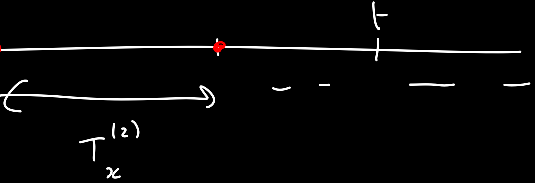
If $(X_n)_{n\geq 0}$ is



$(T_x^{($

a MC starting at x

$P($



(N_t) = number of visits of x

$i)\Big)_i$ iid.

is a Renewal process.

$\longrightarrow$ $\underbrace{\mathbb{P}(T_x^{(1)} \in \mathbb{N}) = 1}_{a=1}$

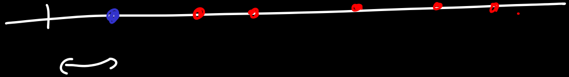
Sketch of proof

Step 1: delayed renewal process.

$\overline{T}_1, \overline{T}_2, \ldots$ independ.

$\overline{T}_1 \sim dG$

$\overline{T}_2, \overline{T}_3, \ldots \sim \underline{dF}$



We can

$\overline{m}(t)$

$\int dF =$

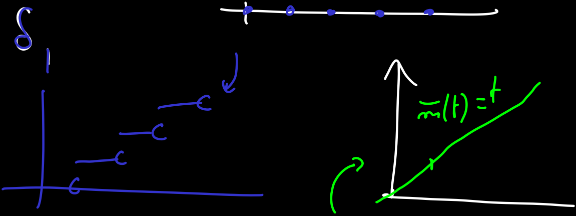
$m(t)$

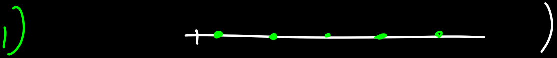
$\overline{T}_1 \sim \mathcal{U}[0,$

choose G s.t.

$$= \mathbb{E}\left[\sum \mathbb{1}_{\bar{T}_1 + \dots + \bar{T}_n \leq t}\right]$$

$$= \frac{t}{\mu}$$

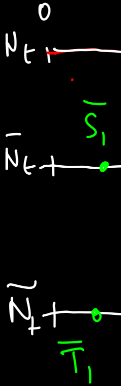




Step 2:

T_1, T

ind



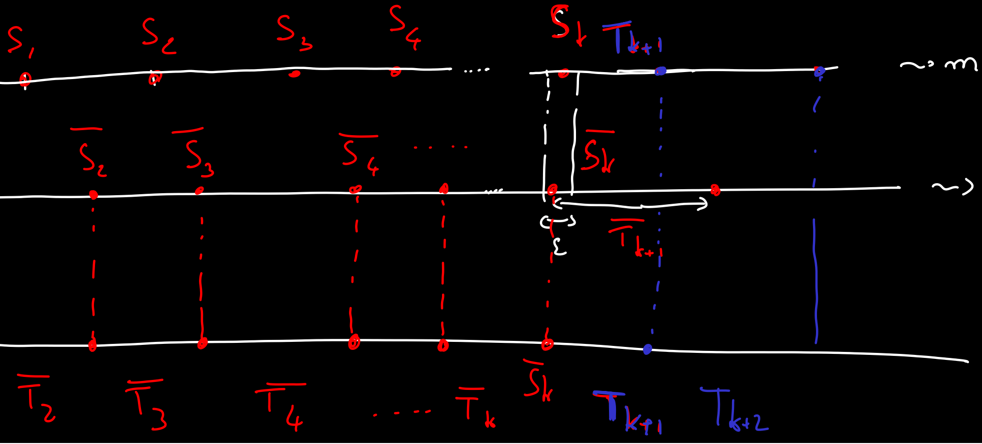
Coupling. Fix $\varepsilon > 0$

$2 , T_3 , \dots$ iid $T_1 \sim dF$

. of $\bar{T}_1 , \bar{T}_2 , \dots$

Claim: $\exists$

consider



$k \geq 1$ s.t.

$|S_k - \bar{S}_k| \leq \varepsilon$

$$\tilde{T}_i = \begin{cases} \bar{T}_i & i \leq k \\ T_i & i > k \end{cases}$$

(t)?

$\bar{m}(t) = \frac{t}{\mu}$

$\rightsquigarrow \tilde{m}(t) = \frac{t}{\mu}$

For t large

$N_{t+h} - N_t$

$\mathbb{E}(N_{t+h}) - \mathbb{E}[N$

$m(t+h) - m(t$

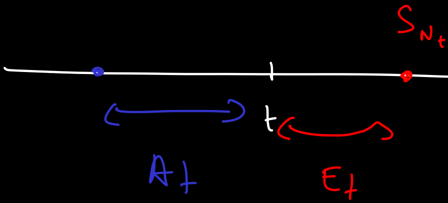
$t \geq k$

$\simeq \tilde{N}_{t+h} - \tilde{N}_t$

$t) = \mathbb{E}(\tilde{N}_{t+h}) - \mathbb{E}(\tilde{N}_t)$

$) \simeq \underbrace{\tilde{m}(t+h) - \tilde{m}(t)}_{\frac{h}{\mu}}$ ∎

$m(t) \simeq \frac{t}{\mu}$ la t large



+1

4. RENEWAL EQUATION.

4.1 Lebesgues-Stieltjes integral & conv

Not: $\mathcal{M} = \{ f : \mathbb{R}_+ \to \mathbb{R}_+$ right continuou
non decreasing

$\mathcal{M}$ = "measures on $\mathbb{R}_+$."

$f \in \mathcal{M} \longrightarrow d\nu = df$ defined by

$$\nu(\,]a,b]\,) = f(b) - f(a)$$

+ extension thm.

olution.

s

$\}$.

→ For all $h \in L^1(df)$

on $\mathbb{R}^+$ $h \geq 0$ meas.

$(h : \mathbb{R}^+ \to \mathbb{R})$

we can define

$$\boxed{\int h \, df}$$

$\uparrow$

Rk: $f(t) = t$ $\qquad \int h \, dt$.

Examples:

- $m \in \mathcal{M}$. $\to$ $\int h \, dm$ can be defined.

- If T is a r.v. on $\mathbb{R}_+$.

  $F_T(t) = \mathbb{P}[T \leq t]$ $\quad F_T \in \mathcal{M}$.

  $dF_T \longrightarrow$ law of T.

Def: Let $G \in \mathcal{M}$. Let

$h : \mathbb{R}_+ \to \mathbb{R}$ s.t.

either $\forall t \int_0^t |h(t-s)| \, dG(s) < \infty$

or $h \geq 0$ a.e.

We define $\forall t \geq 0$

$$\boxed{(h * G)(t) = \int_0^t h(t-s) \, dG(s)}$$

Rk: Let

↳ F_X

$\mathbb{P}\{X +$

X, Y two ind. r.v. on $\mathbb{R}_+$.

$$(t) = \mathbb{P}[x \leq t) \qquad F_Y(t) = \text{--}$$

$$Y \leq t\,] = \int_{s=0}^{t} \underbrace{\mathbb{P}[X + s \leq t)}_{\mathbb{P}(X \leq t-s)} \, dF_Y(s)$$

$$= \int_0^t F_X(t-s)\, dF_Y(s)$$

$$= (F_X * F_Y)(t)$$

$$\boxed{F_{X+Y}}$$

Rk: F

Why is it

$m(t) =$

$=$

$=$

$$\boxed{= F_X * F_Y \, .}$$

$$_{X+Y} = F_Y * F_X \, .$$

useful?

$$\mathbb{E}[N_t)$$

$$\mathbb{E}\Big[\sum_n \mathbb{1}_{T_1 + \dots + T_n \leq t} \Big)$$

$$\sum_n \underbrace{F_{T_1 + \dots + T_n}}_{F^{*n}(t)} (t) \, .$$

$$m(t) = F(t) + F$$

4.2. RENEWAL EQUAT

Def. Let $h : \mathbb{R}_+ \to$

Let $g : \mathbb{R}_+ \to$

We say that g is

renewal equation

$\forall t \geq 0 \quad \big| \, g($

ie. $\quad g =$

$* F(t) + \cdots$

ION. $\quad F(t) = \mathbb{P}[T_1 \le t)$

$\mathbb{R}$ meas. loc. bounded

$\to \mathbb{R}$ s.t. $\forall t \ge 0 \int_0^t |g(t-s)|\, dF(s) < \infty$

a solution of the (h, F)

(ren. eq.) if

$$t) = h(t) + \int_0^t g(t-s)\, dF(s)$$

$h + g * F$

First example:

Prop: m is a solution

(F, F) - renewal eq

$$m = F + m *$$

Pf: $m = \sum_{i \ge 1} F^{*i}$

$= F + \sum_{i \ge 2} F$

$= F + \Big(\sum_{i \ge 1}$

of the

. ie .

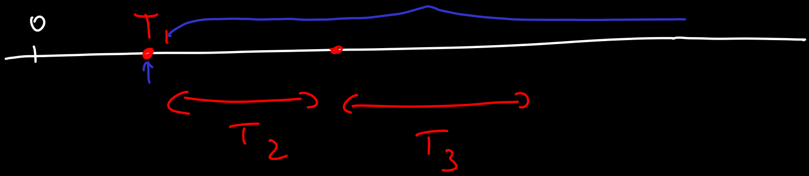
$$\boxed{F\ .}$$

$${}^{*(i-1)} * F$$

properties of $*$

$$F^{*i}\Big) * F = F_f\, m * F$$

Pf. 2:



💡 : condition on T_1 = $N_{t+T_1} - N_{T_1}$ is

$$m(t) = \sum_{n \geq 1} \mathbb{P}[T_1 + \cdots + T_n \leq t]$$

$$= \underbrace{\mathbb{P}[T_1 \leq t)}_{} + \sum_{n \geq 2} \underbrace{\mathbb{P}(T_1 + T_2 + \cdots + T}_{}$$

$$= \int_0^t \underbrace{\mathbb{P}[s + T_2 + \cdots}_{}$$

$$\mathbb{P}[T_2 + \cdots +$$

F

a renewal. process -

$_n \leq t)$

$_{-1} T_n \leq t] \; dF(s)$

$T_n \leq t-s)$

$$m(t) = F(t) + \int_0^t \underbrace{\sum_{n \geq 2} \mathbb{P}[T_2 + \dots + T_n \leq t-s]}_{m(t-s)} \, dF(s)$$

$$= \underbrace{F(t)}_{} + (m * F)(t)$$

$$\sum_{n \geq 2} \underbrace{\mathbb{P}[T_2 + \dots + T_n \leq u]}_{\mathbb{P}[T_1 + \dots + T_{n-1} \leq u)}$$

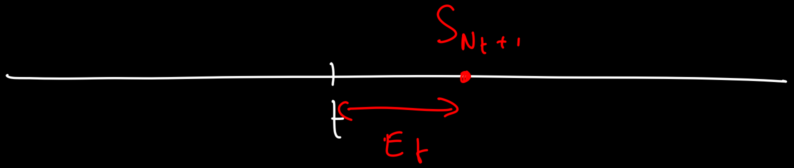
$$= \sum_{n \geq 1} \mathbb{P}(T_1 + \dots + T_n \leq u)$$

$$= m(u)$$

$$\underset{\mathbb{E}[\#\text{ points} \le t]}{m(t)} = \underbrace{F(t)}_{\mathbb{E}[\mathbb{1}_{\text{first} \le t}]} + \underbrace{m * F(t)}_{\mathbb{E}[\#\text{ points af}\ \ \text{the first}}$$

Example 2: excess time.

$$E_t = S_{N_t+1} - t$$

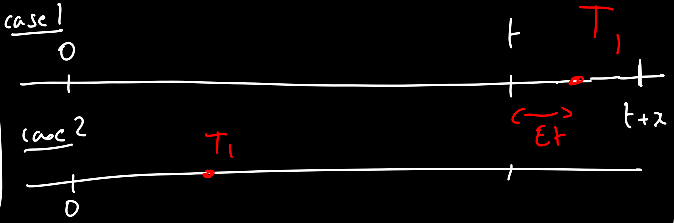


Def: For $x \geq 0$, $e_x(t) = \mathbb{P}[E_t \leq x]$

a

me

What is e_x?



$$\mathbb{P}[E_t \le x] = \underbrace{\mathbb{P}[T_1 > t ,\ E_t \le x]}_{①} + \underbrace{\mathbb{P}[T_1 \le t ,\ E_t \le x]}_{②}$$

① $\mathbb{P}$

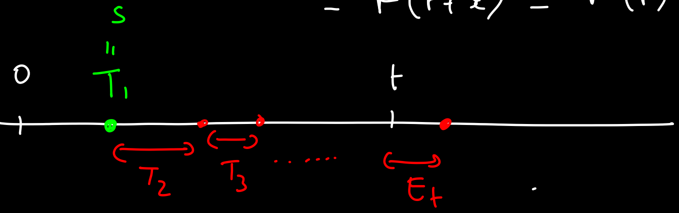
②

$\mathbb{P}[$

"

$$\{T_1 > t\,,\ \underset{\substack{\shortparallel \\ T_1 - t}}{E_t} \le x] = \mathbb{P}[\,T_1 > t\,,\ T_1 \le t + x]$$

$$= F(t+x) - F(t)\ . .$$



$$T_1 \le t\,,\ E_t \le x)$$

$$\int_{s \le} \underbrace{\mathbb{P}[\,E_t \le x \mid T_1 = s\,)}_{= \mathbb{P}[E_{t-s} \le x]}\ \mathbb{P}[T_1 = s)\, ds$$

$$= \int_0^t P(E_{t-s} \le x)\, dF(s)$$

$\mathbb{P}$

observe that E_t is meas. w.r.t

$T_1, T_2, \dots$

$$E_t = \phi_t(T_1, T_2, \dots)$$

$$(T_1 \leq t, E_t \leq x) = \mathbb{P}[T_1 \leq t, \phi_t(T_1, T_2, \dots) \leq x]$$

$$= \underbrace{\int_0^t \mathbb{P}[\underbrace{\phi_t(s, T_2, \dots)}_{\phi_{t-s}(T_2, T_3, \dots)} \leq x]}_{\mathbb{P}[E_{t-s} \leq x]} \, dF(s)$$

$$= \int_0^t e_x(t-s)\, dF(s) = (e_x * F)(t)$$

$e_x(t) = h_x(t) + (e_x * F)(t)$.

where $h_x(t) = F(t+x) - F(t)$

Ccl: e_x is a solution of the (h_x, F)-ren. eq.

$$\boxed{e_x = h_x + e_x * F}$$

$g|[0,t)$ bounded

Exercise:

is a

4.3 WELL-P

EQUATION

Thm: Let

Then there

loc. bounded

g

given by

g

show that $a_z(t) = \mathbb{P}(A_t \le x)$

solution of some (h, F)-r.eq.

OSEDNESS OF THE RENEWAL

$h : \mathbb{R}_+ \to \mathbb{R}$ meas. loc. bounded

exists a unique $g : \mathbb{R}_+ \to \mathbb{R}$ meas.

d , solution of

$= h + g * F$

$$\boxed{= h + h * m}$$

" Trev a

$g = h +$

$= h +$

$= h +$

$= h +$

$\overset{?}{=} h +$

$= h + h *$

s idea"   g solution

$g * F$

$\uparrow$

$(h + g * F) * F$

$h * F + g * F * F .$

$h * F + h * F * F + g * F^{*3}$

$h * F + h * F^{*2} + h * F^{*3} + \dots$

m . . .

Proof: Existence:

Let $g = h + h$

(g is meas.

h is .)

We have

$h + g * F =$

$=$

$=$

$= h$

$* m$.

loc. bounded, because

$$h + (h + h * m) * F$$

$$h + h * F + h * m * F.$$

$$h + h * (F + m * F)$$

$$+ h * m = g.$$

→ g is a sol.

Uniqueness:

g_1, g_2 2 solutions.

$$g_1 - g_2 = (g_1 - g_2) *$$

$$\overset{\text{induction}}{=} (g_1 - g_2) *$$

for every n.

F

$F^{\circ n}$

For every $t \geq 0$

$$|g_1(t) - g_2(t)| = \left| \int_0^t (g_1 - g_2 \right.$$

$$\leq \sup_{[0,t)} \| g_1 - g$$

r

$$) (t-s) dF^{*n}(s) \Big|$$

$$2 \Big| \underbrace{\int_0^t dF^{*n}(s)}$$

$$\underbrace{\mathbb{P}(T_1 + \dots + T_n \leq t)}$$

$$\hookrightarrow 0$$

$$n \to \infty \quad .$$