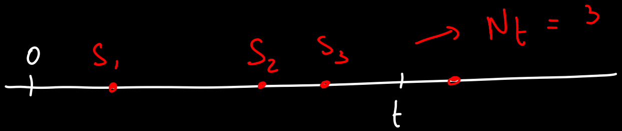
CHAP. 5 (end)

$T_1, T_2, \dots$ iid r.v on $\mathbb{R}^+$

$\underline{F}(t) = \mathbb{P}[T_i \leq t]$ $\quad F(0) < 1$

$\mu = \mathbb{E}[T_1)$ $\quad (\mu \in \mathbb{R}_+ \cup \{+\infty\})$

$S_n = T_1 + \dots + T_n$ “renewal times”



$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{S_n \leq t} \qquad (\ rp(F)\ .)$$

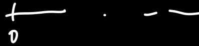
. $m(t) =$

Blackwell

Assume

Then

$\forall h \geq$



$\mathbb{E}[N_t]$ "renewal function"

Thm:

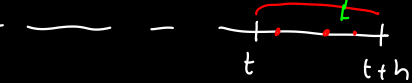
that the law of T_1 is non-arith..

o $$\lim_{t \to \infty} \underbrace{m(t+h) - m(t)} = \frac{h}{\mu}$$

$\mathbb{E}[\underline{N_{t+h} - N_t}]$

renewals in $[t, t+h)$

$\mathbb{E}.[.] \simeq \frac{h}{\mu}$



Renewal. eq. : g

if $\boxed{g =}$

$(g \circ$

$\hookrightarrow \exists!$ sol.

$g = h +$

is a sol of (h, F)-ren. eq.

$$h + g * F .$$

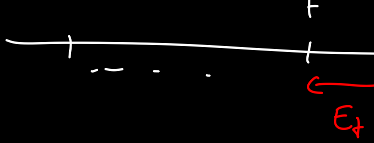
$$F)(t) = \int_0^t g(t-s)\, dF(s)$$

$$h * m$$

g sol. of (h, F)-ren

what is $g(t)$ for t

Motivation



$e_x(t) = P(E_t \le x)$ for t

. eq.

large ?

large

5 ASYMPTOTIC BEHAVIOUR.

Case 1 : $\boxed{h = \mathbb{1}_{[a,b]}}$

g sol. of (h, F) - ren -

$$g(t) = h(t) + \underbrace{\int_0^t h(t-s)\, dm(s)}_{\int_{t-b}^{t-a} dm(s)}$$

$$h(t-s) = \mathbb{1}_{[a,b)}(t-s)$$

$$= \mathbb{1}_{s \in [t-b,\, t-a)}$$

$$g(t) = \underbrace{h(t)}_{\xrightarrow[t\to\infty]{} 0} + \underbrace{m(t-a) - m(t-b)}_{= m(t-b+(b-a)) - m(t-b) \xrightarrow[t\to\infty]{} \frac{b-a}{\mu}}$$

$$\boxed{\lim_{t\to\infty} g(t) = \frac{1}{\gamma} \underbrace{\int_0^\infty h(s)\,ds}_{b-a}}$$

if $h = \mathbb{1}_{[a,b)}$.

How does it generalize?

→ extends to $\sum \lambda_i \mathbb{1}_{I_i}$ ok.

→ extends to dRi functions.

Def.

h

(

an

=

$h : \mathbb{R}_+ \to \underline{\underline{\mathbb{R}_+}}$ meas.

is said to be dRi

directly Riemann integrable) if

$$\underline{\forall \Delta > 0 \quad \sum_{k=0}^{\infty} \sup_{[k\Delta,\, k\Delta + \Delta)} h < \infty}$$

$$\uparrow$$

nd

$$\lim_{\Delta \to 0} \Delta \sum_{k=0}^{\infty} \sup_{[k\Delta,\, k\Delta + \Delta)} h$$

$$\lim_{\Delta \to \infty} \Delta \sum_{k=0}^{\infty} \inf_{[k\Delta,\, k\Delta + \Delta)} h$$

$h : \mathbb{R}_+ \to \mathbb{R}$

$h_+ = \max ($

$h_- = \max$

↳ see notes fo

of h which

not dRi.

is dRi if and only if

$h, 0)$ and

$(-h, 0)$ are dRi.

an example

is integrable but

Prop: Let $h : \mathbb{R}_+ \to$

Assume . h is continu

. there exists

s.t.

$\mathrm{Leb}(\{t : h \text{ is not continuous at } t\}) = 0$

$0 \leq$

and $\int_0^\infty H$

Then h dRi

Pf: see lecture notes

$\mathbb{R}$

uous at a.e. $t \in \mathbb{R}_+$

M non increasing

$|h| \leq M$

$< \infty$.

Sznitman '17

Thm: [Smith key renewal Thm]:

Let h dRi, F non arithmetic

Then $g = h + h * m$ satisfies

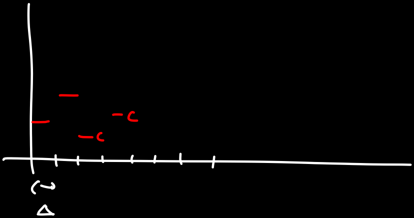
$$\boxed{\lim_{t \to \infty} g(t) = \frac{1}{\mu} \int_0^\infty h(u)\,du}$$

Rk: The case $h = \mathbb{1}_{[0,b)}$ corresponds to Blackwell Thm.

Pf: idea use an approximation of h

by function of the form

$$h_{c,\Delta} = \sum_{k \geq 0} c_k \mathbb{1}_{[k\Delta,(k+1)\Delta)}$$



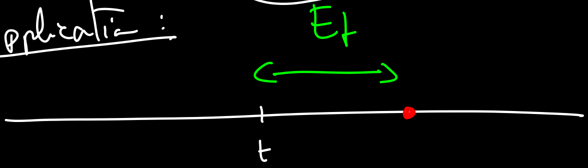
(This can be done because h is dRi)

prove. the result for $h_{c\Delta}$ ...

see notes.

$\boxed{\mu < \infty}$

Application :



$$e_x(t) = \mathbb{P}[E_t \le x]$$

$$\lim_{t \to \infty} e_x(t) \text{ ?}$$

$$e_x = h_x + e_x * F$$

where $h_x(t) = \overbrace{F(t+x)}^{\le 1} - F(t)$
(last week)

Rk: $\mu = \mathbb{E}[T$

$0 \le h_x(t) \le 1 -$

- h_x is continuous at
- $1 - F(t) \searrow$ in
- $\int_0^\infty \mathbb{P}(T_1 > t)\,dt$

h_x is dRi

$$_1) = \int_0^\infty \mathbb{P}(T_1 > t)\,dt$$

$$F(t) = \mathbb{P}(T_1 > t)$$

a.e. (because it is the difference
d of two monotone fcts).

$$= \mathbb{E}(T_1) = \mu < \infty$$

$$\lim_{t \to \infty} \mathbb{P}(E_t \leq x) = \frac{1}{\mu}$$

$$= \frac{1}{\mu} \int_0^\infty \underbrace{\left(F(t+x) - F(\right.}_{\mathbb{E}\{\mathbb{1}_{T_1 \in}}$$

$$= \frac{1}{\mu} \int_0^\infty \mathbb{E}\,[\underbrace{\mathbb{1}_{T_1}}_{\mathbb{1}}$$

$$\int_0^\infty h_x(t)\,dt$$

$$t))\,dt)$$

$$(t, t+x])$$

$$\underbrace{\in (t, t+x)}\,)\,dt$$

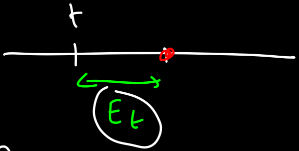
$$t \in [T_1 - x, T_1)$$

$$= \frac{1}{\mu}\,\mathbb{E}\Big[\underbrace{\int_{\max(T_1 - x),0)}^{T_1} dt}\Big)$$

$$= \begin{cases} T_1 & \text{if } T_1 \leq x \\ x & \text{if } T_1 > x \end{cases}$$

$$= \frac{1}{\varphi}\,\mathbb{E}\left[\min(T_1, x)\right]$$

For t large:



$$\mathbb{P}[E_t \le x) \simeq \underbrace{\frac{1}{\mu}\,\mathbb{E}[\min(T_1, x))}_{\in [0,1)}$$

Rk: $\cancel{\mathbb{P}}$ $G(x) = \frac{1}{\mu}\,\mathbb{E}[\min(T_1, x))$

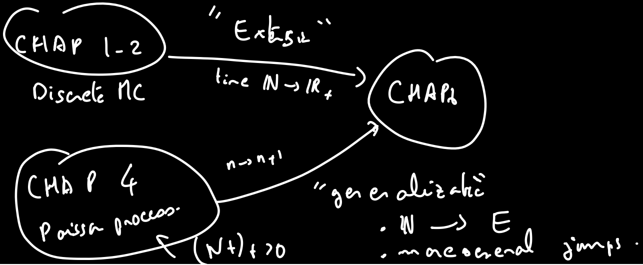
is the delay distribution in the proof of Blackwell's thm.

CHAP. 6 :

CONT. TIME MARKOV CHAINS

Framework: $(\Omega, \mathcal{F}, \mathbb{P})$ proba space

E finite or countable.



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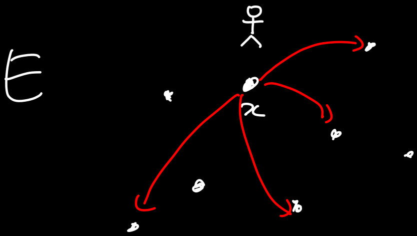


$$\underbrace{n+1}_{\text{time}} = \underbrace{n}_{\text{current time}} + \underbrace{1}_{\text{time step}}$$

continuous time

$$q_{xy} \cdot h \underset{\substack{\uparrow \\ h \text{ small}}}{=} P[\text{jump from } x \text{ to } y \text{ during } [t, t+h))$$

Discrete time MC. (Chap 1.2) $(X_n)_{n \geq 0}$

time : $\mathbb{N} = \{0, 1, 2, \dots$

Initial distribution $X_0 \sim \mu$ $\quad \mu$ proba on E

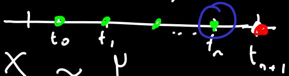
Memoryless property $\mathbb{P}[X_{n+1} = x_{n+1} \mid X_0 = x_0, \dots, X_n = x_n)$

$= \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n)$

Transition probabilities $\mathbb{P}(X_{\boxed{n+1}} = y \mid X_n = x) = \boxed{p_{xy}}$

continuous time MC $(X_t)_{t \geq 0}$

time $\mathbb{R}_+ = [0, \infty)$



$X_0 \sim \mu$.

$\forall\ t_0 \leq t_1 < \dots < t_{n+1}$

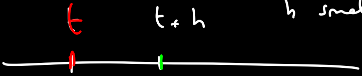
$$\mathbb{P}[X_{t_{n+1}} = x_{n+1} \mid X_{t_0} = x_0, \dots, X_{t_n} = x_n)$$

$$= \mathbb{P}[X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n)$$

μ-scopic generator: $x \neq y$

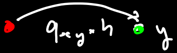
$$\mathbb{P}[X_{t+h} = y / X_t = x)$$

$$= q_{xy} \times h + o(h)$$



h small.

$X_t = x$ $X_{t+h} = y$



if h is small

$$\mathbb{P}[X_{t+h} = x \mid X_t = x) = 1$$

.1 DEF VIA GENERATOR

E finite or countable

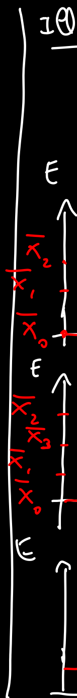
Def. (jump process).

Let $X = (X_t)_{t \geq 0}$ be a cont. time stochastic process with values in E.

We say that X is a jump process without explosion if a.s. :

(i) $t \longmapsto X_t$ is right continuous.

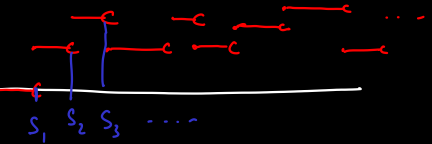
(ii) $\forall t > 0$ the number of discontinuity points of $s \to X_s$ inside $[0, t)$ is finite

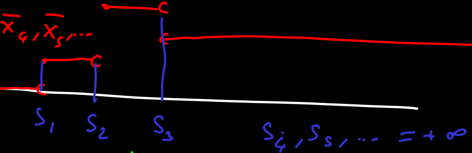


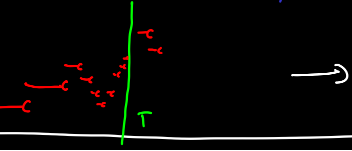
ushalic (ex: Right continuity

$$\boxed{\forall t} \; \exists \varepsilon > 0 \text{ s.t. } \quad \forall\, 0 \leq h \leq \varepsilon$$

$$X_{t+h} = X_t \quad)$$







jump process with explosion

(i) is satisfied

(ii) is not satisfied.

jump times:

$S_0 = 0$

$S_{i+1} =$

(condition (ii) im

skeleton

$\forall n \in \mathbb{N} \quad \bar{X}_n := X$

if $\exists\, n_0$ s.t.

$\forall n \geq n_0 \quad X_n = X_{n_0}$

$(\inf \emptyset = \infty)$

$\inf \{ t >^{S_i} \text{ s.t. } X_t = X_{S_i} \}$

plies that $S_n \xrightarrow[n\to\infty]{} \infty$)
a.s.

S_n if $S_n < \infty$

$S_n = +\infty \quad \forall n \geq n_0.$

$_0 - 1$

Def: A generator (a Q

a family $q = (q_{xy})_{x,y \in E}$

- $q_{xy} \geq 0 \quad x \neq y$
- $\forall x \sum_{y \neq x} q_{xy} < \infty$
- $q_{xx} = - \sum_{y \neq x} q_{xy}$.

$\Updownarrow$

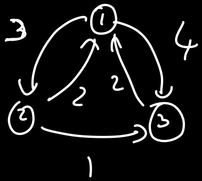
Rk: $\forall x \sum_{y \in E} q_{xy} = 0$

- matrix) is

where

weighted graph representation.

$E = \{1, 2, 3\}$ $Q = \begin{pmatrix} -7 & 3 & 4 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{array}{l} \to \\ \to \\ \to \end{array}$



q_{xy} jump rate from x to y

$x \neq y$ $\mathbb{P}[X_{t+h} = y \mid X_t = x] = q_{xy} \cdot h + \underset{h \to 0}{o}(h)$

$x = y$ $\mathbb{P}[X_{t+h} = x \mid X_t = x] = 1 - \left(\sum_{y \neq x} q_{xy}\right) \cdot h + o(h)$

Notation: $\cdot\ q(x) = q_{xx} = -\sum_{y \neq x} q_{xy}.$

$$\cdot\ \delta_{xy} = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}$$

"continuous time Markov without explosion with initial distribution μ and generator q

$= 1 + q_{xx} h + o(h)$

Def: Let X

Let μ be a

We say

if: (i) X_0

(ii) $\forall t,$

$\mathbb{P}[X_{t_{n+1}} =$

(iii) $\forall x,$

$\mathbb{P}(X_{t+h}$

uniformly

be a jump process without explosi

distribution on E, q a generator.

that X is a $\mathrm{CTMC}(\mu, q)$

$\sim \mu$

$< \dots < t_{n+1} \quad \forall x_1, \dots, x_{n+1} \in E$

$x_{n+1} \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n) = \mathbb{P}[X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n)$

$y \in E \ \forall t > 0$ as $h \searrow 0$

$= y \mid X_t = x] = \delta_{xy} + q_{xy} \cdot h + o(h)$

in $t \geq 0$ $y \in E$.

Rk: (iii) $\forall x$

s.t. φ_x

and $\forall h > 0$

$\mathbb{P}[X_{t+h} = y \mid X_t$

where $0 \leq \varphi$

$\exists \varphi_x : \mathbb{R}_+ \to \mathbb{R}$

$(h) \xrightarrow[h \searrow 0]{} 0$

$\forall y \in E$

$$=x] = \begin{cases} 1 - q(x).h + h\varphi_{x,x,t}(h) \\ q_{xy}.h + h\,\varphi_{x,y,t}(h) \end{cases}$$

$${}_{x,z,t}(h) \le \boxed{\varphi_x(h)}.$$

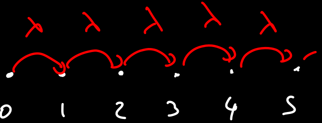
Ex: Poisson process: (Cha

Let $(N_t)_{t \geq 0}$ be a pp(λ)

Then N is a CTMC $(\mu$

where $\mu = \delta_0$

$q = (q_{xy})_{x,y \in \mathbb{N}}$ $\quad q_{xy} =$



q

.3).

.

, q)

$$\begin{cases} \lambda & y = x+1 \\ -\lambda & y = x \\ 0 & \end{cases}$$

- $N_0 = 0$ a.s.
- follows from MP of pp(λ)
- $\mathbb{P}[N_{t+h} = y \mid N_t = x)$

$$= \frac{\mathbb{P}[N_t = x, N_{t+h} = y)}{\mathbb{P}(N_t = x)}$$

$$= \frac{\cancel{\mathbb{P}[N_t = x]}\,\mathbb{P}[N_{t+h} - N_t = y - x)}{\cancel{\mathbb{P}[N_t = x)}}$$

$$= \mathbb{P}[N_{t+h} - N_t = y - x)$$

$$= \mathbb{P}[N_h = y - x) = \begin{cases} \overset{1_{x=y}\,\downarrow}{} - \lambda \\ \lambda h \\ 0 \cdot h \end{cases}$$

$\boxed{Q}$ Does CTMC (γ, q) exist?

q_{xx}

$h + o(h)$ if $x = y$

$+ o(h)$ $\quad y = x+1$

$+ o(h)$ $\quad y \notin \{x, x+1\}$

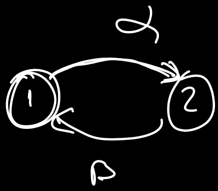
2 NON RIGOROUS SECTION:

CONSTRUCTIVE APPROACH ON EXAMPLES

Example 1: 2 state MC.

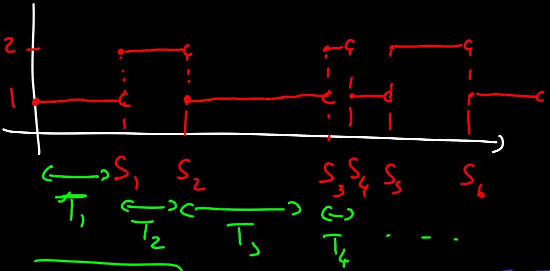
$E = \{1, 2\}$ $\qquad q = \begin{pmatrix} -\alpha, & \alpha \\ \beta & -\beta \end{pmatrix}$

$\alpha, \beta > 0$



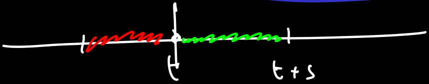
$(X_t)_{t \geq 0}$ $\quad X_t \sim CTMC(\delta_1, q)$?

$X_0 = 1$



$\boxed{T_1 ?}$

$\boxed{T_1 \sim Exp(\alpha)}$



$P[X_h$

$P[T_1$

T_1 mo

$P[T_1 >$

$= P[\forall u$

$\overset{MP}{=} P[\forall u$

$$= 2 \mid X_0 = 1) = \underset{> 0}{\alpha} h + o(h)$$

$$< h) \geq \underbrace{\alpha h}_{} + o(h)$$

n arithmetic ...

$$\underline{t + s \mid T_1 > t)}$$

$$\in [t, t+s)\ X_u = 1 \mid \forall \underset{\uparrow}{u} \leq t\ X_u = 1]$$

$$\leq s\ \ X_u = 1) = \underline{\mathbb{P}(T_1 > s]}$$

$$T_1 \sim \mathrm{Exp}(\underset{\uparrow}{\lambda})$$

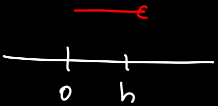
What is λ?

$$\mathbb{P}(T_1 \geq h) = \mathbb{P}$$

$$\underset{\uparrow}{=} \mathbb{P}$$

to justify.

$$= 1$$



$[\ \forall u \leq h \ X_u = 1)$

$[X_h = 1] + o(h)$

$- \alpha h + o(h)$

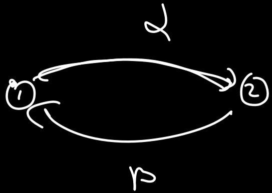
$T_1 \sim \mathrm{Exp}(\lambda)$

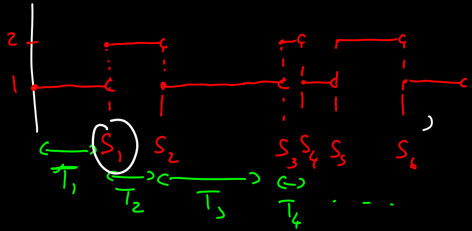
$P[T_1 \geq h)$

$= e^{-\lambda h}.$

$= 1 - \lambda h + o(h)$

$$\boxed{\lambda = \alpha}$$





$T_1 \sim \text{Exp}(\alpha)$

$T_2 \sim \text{Exp}(\beta)$

$T_3 \sim \text{Exp}(\alpha)$

$T_4 \sim \text{Exp}($

T_i are indep.

$T_{2i+1} \sim \text{Exp}(\alpha$

$)$ indep.

$\beta)$

$) \quad T_{2i} \sim Exp(\beta)$

Candidates

$X_t = \begin{cases} 1 & \text{is } t \in [\\ 2 & \text{i } t \in \end{cases}$

$S_i = T_1 + \dots +$

$\overline{X_n} = \begin{cases} 1 & \text{if} \\ 2 & \text{if} \end{cases}$

$s_i, S_{i+1})$

(S_{i+1}, S_{i+2})

T_i

n is even

n is odd