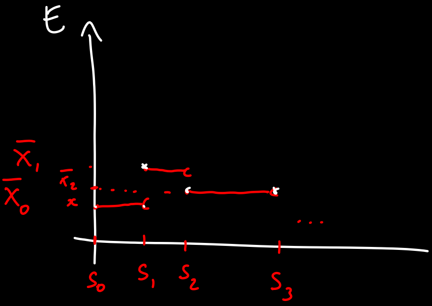
5. CONTINUOUS-TIME MC

$(\Omega, \mathcal{F}, \mathbb{P})$ E finite or countable

jump process $X = (X_t)_{t \geq 0}$

a.s (i) $t \mapsto X_t$ right cnt.

(ii) $\forall t$ $\#\{\text{jumps in } [0,t]\} < \infty$


E
$\bar{X}_1$
$\bar{x}_2$
$\bar{X}_0$
x
...
S_0
S_1
S_2
S_3

jump times : $S_0, S_1, S_2, \ldots$

skeleton : $\bar{X}_n = X_{S_n}$

$q_{xy} \geq 0$
$x \neq y$

$q_{xx} \leq 0$

$\sum_y q_{xy} = 0$

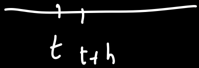
$q(x) = -q_{xx}$

CTMC $\{\gamma, q\}$ $\quad$ $(X$

(i) $X_0 \sim \rho$

(ii) $\mathbb{P}[X_{t_{n+1}} = x_{n+1} |$

(iii) $\mathbb{P}[X_{t+h} = y |$



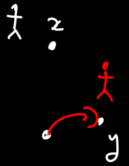
$t)_{t>0}$

$$q(x) = \sum_{y \neq x} q_{xy}$$

$X_{t_0} = x_0, \dots, X_{t_n} = x_n) = \mathbb{P}\left(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n\right)$

$$X_t = x) = \begin{cases} q_{xy} \cdot h + o(h) & x \neq y \\ \underline{1 - q(x)} \cdot h + o(h) & x = y \end{cases}$$

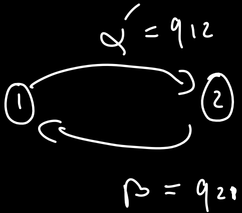
$t+h$

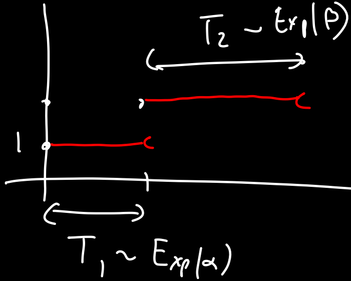


$1 - q(x)\, h$

$q_{xy} \cdot h$

$E = \{1, 2\}$

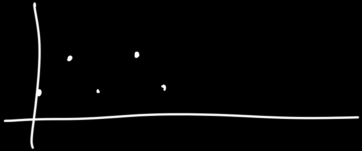

$\alpha = q_{12}$
1
2
$\beta = q_{21}$


$T_2 \sim Exp(\beta)$
1
$T_1 \sim Exp(\alpha)$

$$\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

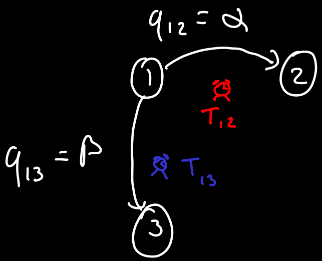
In general

q_{xy} ⟶ " parameter for the time to jump fr
$x \to y$ "



m

3 STATE - MC.


$q_{12} = \alpha$
1
2
T_{12}
$q_{13} = \beta$
T_{13}
3

Start at

$X_0 = 1$

- $T_{12} \sim \text{Exp}(\alpha)$
- $T_{13} \sim \text{Exp}(\beta)$

indep.

jump

$T_1 \sim$

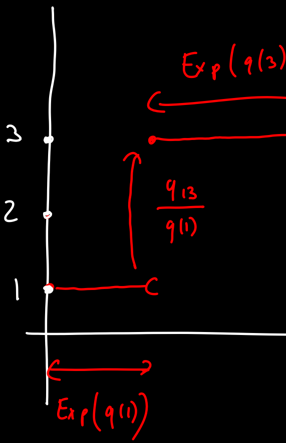
$\mathbb{P}\{\text{jump } 1$

at $T_1 = \min(T_{12}, T_{13})$

$$\mathrm{Exp}(\underbrace{\alpha+\beta}_{q_{12}+q_{13} = q(1)})$$

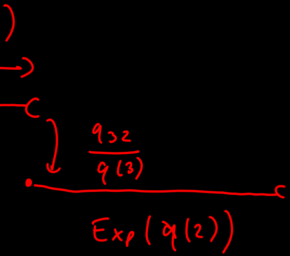
$$\to 2) = \mathbb{P}[T_1 = T_{12}]$$

$$= \frac{\alpha}{\alpha+\beta} = \frac{q_{12}}{q(1)}$$


Exp(q(3)
3
2
1
$\frac{q_{13}}{q(1)}$
Exp(q(1))

Skeleton $\overline{X}_0$

↳ $(\overline{X}_n)_{n \geq 0}$


$\frac{q_{32}}{q(3)}$
$\text{Exp}(q(2))$

$= 1 \ , \ \overline{X}_1 = 3 \ , \ \overline{X}_2 = 2 \ \ldots$

Discrete time MC

transition $\quad K_{xy} = \frac{q_{xy}}{q(x)}$

3. DEF BY SKELETON
TIME

q : fixed generator

3.1 DISCRETE CHAIN AS

Def: Let $x, y \in E$

$$\begin{cases} \text{If } q(x) > 0 & K_{xy} \\ \text{If } q(x) = 0 & K_{xy} \end{cases}$$

E HOLDING

JUGATED TO Q

$$= \begin{cases} \dfrac{q_{xy}}{q(x)} & x \neq y \\ 0 & x = y \end{cases}$$

$$= \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}$$

Rk: K transition probability.

if $q(x) > 0$

$$\sum_y K_{xy} = \sum_{y \neq x} \frac{q_{xy}}{q(x)} = \frac{q(x)}{q(x)} = 1$$

$q(x) = 0$

$$\sum_y K_{xy} = K_{xx} = 1$$

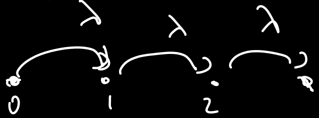
Ex:

- $PP(\lambda)$

$q(n) = \lambda$

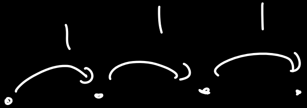
$q_{x\,x+1} = \lambda$

$q =$

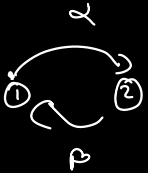


q

K

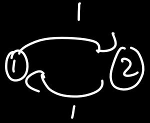


- q

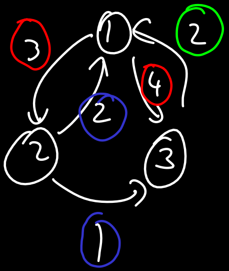


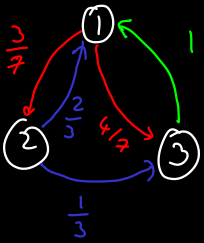
K

K



$E = \{1, 2, 3\}$




1
2
3
3/7
1
2/3
4/7
1/3

$q(1) = 7$

$q(2) = 3$

$q(3) = 2$

3.2. SOMETHING C

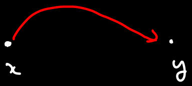
....

μ prob a measure

q generator

goal : define $(X_t$

$\mathbb{P}[X_{t+h} = y |$



AN GO WRONG

on E

$)_t$ $\text{CTMC}(\nu, q)$

$X_t = x) \simeq q_{xy} \cdot h + o(h)$

Let $Y = (Y_n)$ be a MC $(\nu,$

P

discret

Let $M_1, M_2, \ldots$ iid $\exp(1)$

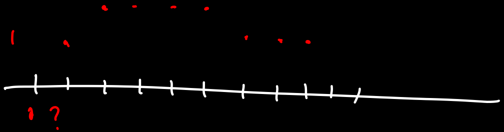
Set $T_i := \frac{1}{q(Y_i)} \times M_i$

Cnd. on Y

$$T_i \sim \text{Exp}(q(Y_i))$$

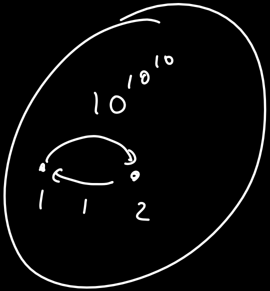
and there are independent.

k)



$\tilde{X}_n = X_{h \times n}$

$h > 0$



Let $Y = (Y_n)$ be a $\underset{\substack{\uparrow \\ \text{discret}}}{\text{MC}}\,(\underset{\uparrow}{\mu}, \underset{\uparrow}{K})$

Let $H_1, H_2, \dots$ iid $\exp(1)$

Set $T_i := \frac{1}{q(Y_i)} \times H_i$

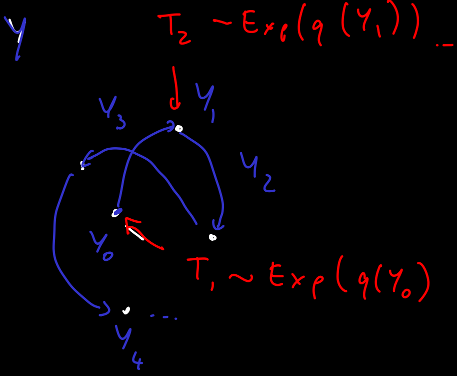
Cnd. on Y

$$T_i \sim \text{Exp}(q(Y_i))$$

and there are independent.

$$\Big[\ \mathbb{P}[\underbrace{T_i > s}_{} \mid Y] = \mathbb{P}[H_i > \underbrace{q(Y_i)}_{}.s \mid Y]$$

$$= e^{-q(Y_i).s}\ \Big]$$


Y
$T_2 \sim \mathrm{Exp}(q(Y_1))$ ...
Y_1
Y_3
Y_2
Y_0
$T_1 \sim \mathrm{Exp}(q(Y_0))$
...
Y_4

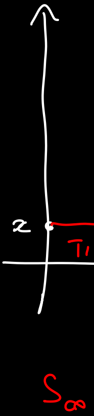
E

$$S_i = T_1 + T_2 + \cdots + T_i \qquad i \geq 1$$

Define $X_t = Y_n$ if $t \in [S_n, S_{n+1})$

We

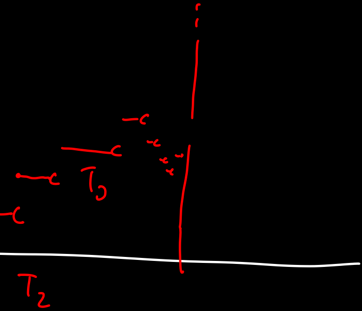
....



have define X_t for every $t \geq 0$.

Really ?

Not really ..



$= \lim S_n$ could be finite!

Def:

Rk:

Q1 . D

We say that q has no explosion if $\forall$ choice of ν

$$S_\infty \left(= \lim_{n\to\infty} S_n\right) = +\infty \quad a.s.$$

It is (only) a condition on q.

oes there exist q with explosion.

↳ Sect 3.3

If q has no explosion is $(X_t)_{t\geq 0}$ a CTMC(ν, q)? → Sect. 3.4

3.3

E

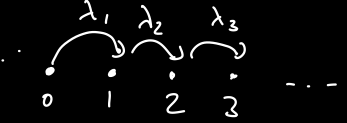
q

lc

y

BIRTH CHAIN

$= \mathbb{N}$. fix $(\lambda_i)_{i \geq 1}$

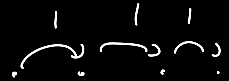


$q_{i\,i+1} = \lambda_i$

$q_{ii} = -\lambda_i$

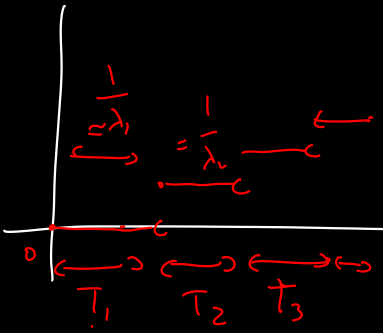
$q_{ij} = 0$ if $j \in \{i, i+1\}$

=



$n = n$

$T_i \sim \text{Exp}(\lambda_i)$



$$S_\infty = \sum_{i=1}^{\infty} T_i$$

is $S_\infty < \infty$ or

$S_\infty = +\infty$ a.s.

Rk: $PP(\lambda) \to$ birth chain with $\lambda_i = \lambda$ $\forall i$.

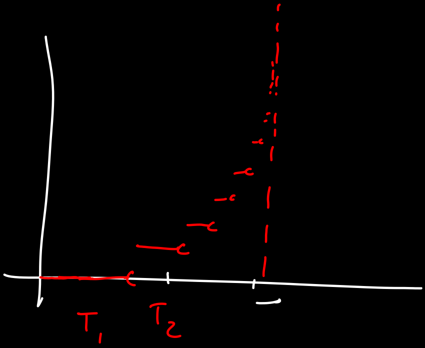
Thm: (The birth chain q has no explosion)

$$\Updownarrow$$

$$\left(\sum_{i \geq 1} \frac{1}{\lambda_i} = +\infty \right)$$

λ

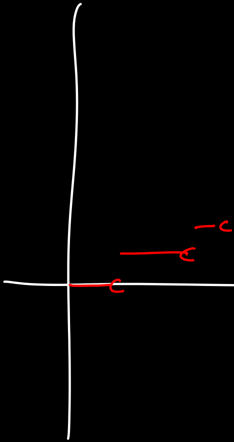
$\lambda_i = \frac{1}{2^i}$

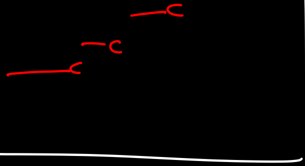


$S_\infty < \infty$ a.s.

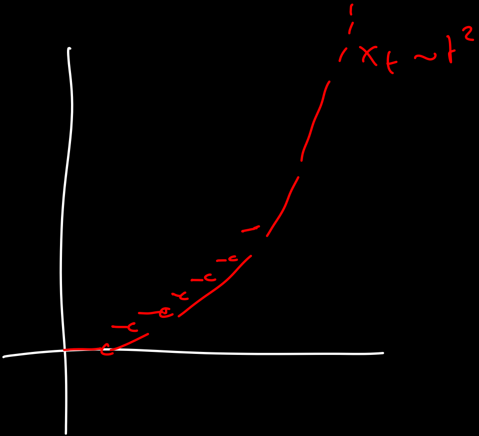
~~T~~ = $\tilde{T}_i = \lambda_i T_i$

$\lambda_i = \lambda$





$$\lambda_i = \frac{1}{N_i}$$



Proof,

$$\boxed{\Downarrow}$$

$$S_\infty = \sum_{i=1}^{\infty} T_i \quad \left(S_\infty \in \mathbb{R}_+ \cup \{+\infty\} \right)$$

If $\sum \frac{1}{\lambda_i} < \infty$

$$\mathbb{E}\left[S_\infty \right] = \mathbb{E}\left(\sum_{i=1}^{\infty} T_i \right) = \sum_{i=1}^{\infty} \mathbb{E}\left(T_i \right)$$

$$= \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$$

Hence (since $S_\infty \geq 0$)

$S_\infty < \infty$ a.s.

$\boxed{\Uparrow}$ Assume $\sum \frac{1}{\lambda_i} = +\infty$.

$\mathbb{E}[\underbrace{e^{-S_\infty}}_{\in \mathbb{R}^+}]$ $\quad S_\infty \in \mathbb{R}_+ \cup \{+\infty\}$

$e^{-\infty} = 0$

$$\mathbb{E}[e^{-S_\infty}] \underset{\uparrow}{=} \lim_{n\to\infty} \mathbb{E}[e^{-S_n}]$$

monotone cv

$S_n \uparrow S_\infty$ a.s.

$\mathbb{E}[$

(a

$\mathbb{E}$

$$e^{-S_n}\Big) = \mathbb{E}\Big[e^{-T_1} \cdot e^{-T_2} \dots e^{-T_n}\Big)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{1+\frac{1}{\lambda_i}} \right)$$

T_i are indep.

$$\xrightarrow[n\to\infty]{} 0 \quad \left(\text{because } \sum \frac{1}{\lambda_i} = +\infty\right)$$

$$\geq 0 \qquad \prod (1+a_i) = +\infty \iff \sum a_i = +\infty\Big)$$

$$\Big[\underbrace{e^{-S_\infty}}_{\geq 0}\Big) = 0 \quad \Rightarrow e^{-S_\infty} = 0 \text{ a.s.}$$

$$\Rightarrow S_\infty = +\infty \text{ a.s.} \quad \blacksquare$$

3.4:

$q \quad g$

$\rightarrow (1$

TR_1

$Y=$

$(q$

NON-EXPLOSION CHARACTERIZATION

enerator on E.

$\left(c_{xy} = \frac{q_{xy}}{q(x)} \right)$

For $x \in E$, let

$(Y_n^{(x)})_{n \geq 0}$ be a $MC(p, K)$

has no explosion $) \iff \left(\forall x \quad \underline{\sum_{n \geq 0} \frac{1}{q(Y_n^{(x)})}} = +\infty \text{ a.s.} \right)$

p

Rk: $\sum_{n \geq 0} \frac{1}{\underbrace{q(Y_n)}}$ is random.

Proof: condition on Y

$T_1 \sim \mathrm{Exp}(q(Y_1))$

$T_2 \sim \mathrm{Exp}(q(Y_2))$

$\vdots$

$\sum_{n \geq} \frac{1}{q(Y_n)}$

(see notes)

Appr

q

•

•

•

lication (sufficient condition for non explosion)

is non explosive if

- E finite (eg 2-state
  3-state)

- $\sup_{x \in E} q(x) < \infty$ (ex. Poisson)

- The chain K is irreducible and recurrent.

3.5 key

Thm: [C

- Let

process

- Let q

Then

equiv

Thm:

haracterization of $\mathrm{CTMC}(\varphi, q)$]

$X = (X_t)_{t \geq 0}$ be a jump

without explosion.

be a generator with no explosion

The following are

valent :

- X

- The

is

and

hold

and

is $CTMC(\nu, q)$

skeleton $Y = \bar{X}$ of X

a (discrete) $MC(\nu, K)$

condition on Y, the

ing times satisfy

$S_i - S_{i-1} \sim Exp(q(Y_i))$

are independent.

Pf: admitted

- Consequences.
  - Existence of CTMC
  - uniqueness of the

    (if X, Y are

    $\forall\, t_1 < \dots < t_n$

    $(X_{t_1}, \dots, X_{t_n})$
  - constructive alg

    ↳ see Norris

for non explosive q.

Law of a CTMC(ν, q)

CTMC(ν, q)

$\sim \left(Y_{t_1}, \dots, Y_{t_n} \right) \Big)$

rithms.

4. MARKOV PROPERTIES.

q non explosive.

Framework: $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in E})$ (X_t)

s.t. under $\mathbb{P}_x$, X is C

(such $(\mathbb{P}_x)_x$ exist. take μ $\mu(x) > 0$

Consider $(X_t)_{t \geq 0}$ CTMC (μ, q)

Then let $\mathbb{P}_x = \mathbb{P}(\cdot \mid X_0 = x)$)

$t \geq 0$

$\mathrm{TMC}(\nu, q)$

$\forall x$.

SiMP: Fix $t \geq 0$

$x \in E$

Conditionally on $X_t = x$

$(X_{t+s})_{s \geq 0}$ is $\mathrm{CTMC}(\delta_x, q)$

independent of

$(X_u)_{u \leq t}$.

Strong Markov:

the same applies if we
replace t by a random stopping time T.

5) TRANSITION PROBABILITIES

$(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in E})$ $\underline{q \text{ generator}}$

$X = (X_t)_{t \geq 0}$ is $CTMC(\delta_x, q)$

under $\mathbb{P}_x$

Define : for $t \geq 0$ for $x, y \in E$

$$\boxed{p_{xy}(t) = \mathbb{P}_x[X_t = y)}$$

discrete

$$p^{(n)}_{xy} \longleftrightarrow p_{xy}(t)$$

Rks:

- $\forall t \geq 0$ $\quad (p_{xy}(t))_{x,y\in E}$ is a transition proba.

$$\sum_y p_{xy}(t) = \sum_y \mathbb{P}_x(X_t = y) = 1$$

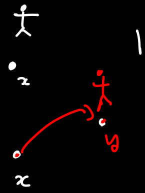
- $\forall x \quad p_{xx}(t) \geq e^{-q(x)t} \quad \forall t$

- $\forall x, y \in E$

$$p_{xx}(h) = 1 - q(x)\cdot h + o(h)$$

$$p_{xy}(h) = q_{xy}\cdot h + o(h) \quad (x \neq y)$$

h small



$-q(x).h$

$q_{xy}.h.$

Prop: (CK equation) (semi-group property)

$$\forall\, t, s \geq 0$$

$$p_{xy}(t+s) = \sum_z p_{xz}(t)\, p_{zy}(s)$$

Pf: $\mathbb{P}_x[X_{t+s}=y) = \sum_z \underbrace{\mathbb{P}_x[X_{t+s}=y \mid X_t=z)} \; \mathbb{P}_x[X_t=z]$

$$\underset{\substack{\uparrow \\ \text{SiMP}}}{=} \mathbb{P}_z[X_s=y) \qquad \boxtimes$$

knowing Q, what is $p_{xy}(t)$?

$$p_{xx}(h) = \overset{= p_{xx}(0)}{1} - q(x)\cdot h + o(h)$$

$$p_{xy}(h) = q_{xy}\cdot h + o(h) \quad (x \neq y)$$

$$p_{xx}'(0) = \lim_{h \to 0} \frac{p_{xx}(h) - p_{xx}(0)}{h}$$

$$= -q(x)$$

$$x \neq y \quad p_{xy}'(0) = q_{xy}$$

Thm: $\forall x, y \in E$ $\quad p_{xy}$ is $\mathcal{C}^1$ on $\mathbb{R}_+$

and $\forall t \geq 0$

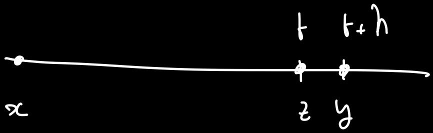
$$p'_{xy}(t) = \left(\sum_{z \neq x} q_{xy}\, p_{zy}(t)\right) \underbrace{-q(x)}_{+\,q_{xx}} p_{xy}(t)$$

[Backward equation]

$$p'_{xy}(t) = \underline{\left(\sum_{z \neq y} \underline{p_{xz}}(t)\, q_{zy}\right)} - \underline{p_{xy}(t)\, q(y)}$$

(forward equation)

Forward equation



Pf: E finite (forward eq.)

$$p_{xy}(t+h) \overset{CK}{=} \sum_{z} p_{xz}(t)\, p_{zy}(h)$$

$$= \underbrace{p_{xy}(t)}_{} \times \underbrace{p_{yy}(h)}_{1-q(y)h+o(h)} + \sum_{z \neq y} p_{xz}(t)\, \underbrace{p_{zy}(h)}_{q_{zy}\cdot h+o(h)}$$

p_{xy}

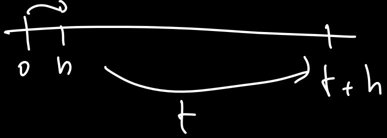
$\rightarrow$

B

$$(t+h) \underset{\substack{\uparrow \\ E\ \text{finite}}}{=} p_{xy}(t) + \left(\sum_{z \neq y} p_{xz}(t)\, q_{zy} - p_{xy}(t)\, q(y) \right) h + o(h)$$

$$\boxed{p_{xy}'(t) = \left(\sum_{z \neq y} p_{xz}(t)\, q_{zy} \right) - p_{xy}(t)\, q(y)}$$

ack ward : some proof



E finite: $E \{1, \dots, k\}$

$$P(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1k}(t) \\ \vdots & & \vdots \\ p_{k1} & \dots & p_{kk}(t) \end{pmatrix}$$

$$Q = \begin{pmatrix} q_{11} & \dots & q_{1k} \\ \vdots & & \vdots \\ q_{k1} & \dots & q_{kk} \end{pmatrix}$$

$$p_{xy}'(t) = \sum_{z \in E} q_{xz} \, p_{zy}(t) \rightarrow P'(t) = Q \times$$

$$P'(t) = P$$

Thm: If E is finit, we have

$$\forall t \geq 0 \quad \boxed{P(t) = e^{tQ}}.$$

$P(t)$ (Backward

$(t) \times Q.$