Introduction to Mathematical Finance
Dylan Possamaï

Assignment 12

On utility maximisation

Consider a general arbitrage-free single-period market, where the interest rate is taken to be 0. Fix $x > 0$ and let $U : [0, \infty) \longrightarrow \mathbb{R}$ be a concave, increasing (utility) function, continuously differentiable on $(0, \infty)$, such that

$$\sup_{\xi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[U\big(x + \xi \cdot (S_1 - S_0)\big)\big] < \infty, \tag{0.1}$$

with

$$\mathcal{A}(x) := \big\{\xi \in \mathbb{R}^d : x + \xi \cdot (S_1 - S_0) \geq 0,\ \mathbb{P}\text{–a.s.}\big\}.$$

Furthermore, assume that the supremum is attained in an interior point $\xi^\star$ of $\mathcal{A}(x)$.

1) Show that

$$U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)|S_1 - S_0| \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P}),$$

and the *first order condition*

$$\mathbb{E}^{\mathbb{P}}\big[U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)(S_1 - S_0)\big] = 0.$$

2) Show that $\mathbb{Q}$ given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \frac{U'(x + \xi^\star \cdot (S_1 - S_0))}{\mathbb{E}^{\mathbb{P}}[U'(x + \xi^\star \cdot (S_1 - S_0))]},$$

is a risk-neutral measure.

Why do we need super-martingale deflators?

We put ourselves in the setting of the duality approach to utility maximisation. The goal of this exercise is to illustrate why we need, for fixed $z > 0$, to work with the larger set $\mathcal{Z}(z)$ in the dual problem $j(z)$, instead of the set $z\mathcal{M}(S)$ of densities of risk-neutral measures (multiplied by z).

We construct a one-period market defined on a *countable* probability space Ω. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers such that

$$\sum_{n=0}^{\infty} p_n = 1, \text{ and } \lim_{n\to+\infty} p_n = 0.$$

We also let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers starting at $x_0 = 2$ and also decreasing to 0 and n goes to $+\infty$, but less fast than $(p_n)_{n\in\mathbb{N}}$, in a sense to be made precise later on.

Finally, we consider a market with constant non-risky asset and one risky asset, starting from $S_0 = 1$, and such that the $\mathbb{P}$-distribution of S_1 is given by

$$\mathbb{P}[S_1 = x_n] = p_n,\ n \in \mathbb{N}.$$

We equip the probability space with the natural filtration of S.

1) Show that the market is arbitrage free, and argue that $\mathcal{M}(S) \neq \emptyset$. Is the market complete?

2) Determine an interval $[a, b] \subset \mathbb{R}$ such that $\mathcal{V}(1) = [a, b]$, that is to say $X_1^{1,\Delta} \geq 0$, $\mathbb{P}$–almost surely, if and only if $\Delta \in [a, b]$.

3) Assume for this question and all the remaining ones that the following series (whose terms are negative for n large enough) are well-defined

$$\sum_{n\in\mathbb{N}} p_n \log(x_n),\ \sum_{n\in\mathbb{N}} p_n \frac{x_n-1}{x_n},$$

and that

$$\sum_{n\in\mathbb{N}} p_n \frac{x_n-1}{x_n} > 0.$$

Maximise the function

$$f(\Delta) := \mathbb{E}^{\mathbb{P}}\big[\log(X_1^{1,\Delta})\big],$$

over $[a,b]$. Derive the optimal investment $\Delta^\star \in \mathcal{V}(1)$.

4) Compute explicitly (in terms of $(x_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$) the value function $v(x)$ for logarithmic utility. Show that $v'(1)=1$.

5) Compute the corresponding dual optimiser $Z^\star \in \mathcal{Z}(1)$.

6) Assume now that

$$\sum_{n\in\mathbb{N}} \frac{p_n}{x_n} < 1.$$

Conclude that $Z^\star \in \mathcal{Z}(1)$ is not a martingale, but only a super-martingale. In particular, $Z^\star$ is not the density process of a martingale measure for the process S, and hence the infimum

$$\inf_{\mathbb{Q}\in\mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}\left[J\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right],$$

is not attained.

7) Provide an example of $(x_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ such that the following series are well-defined

$$\sum_{n\in\mathbb{N}} p_n \log(x_n),\ \sum_{n\in\mathbb{N}} p_n \frac{x_n-1}{x_n},$$

and such that

$$\sum_{n\in\mathbb{N}} p_n \frac{x_n-1}{x_n} > 0,\ \sum_{n\in\mathbb{N}} \frac{p_n}{x_n} < 1.$$