Introduction to Mathematical Finance
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Assignment 2

Cross-hedging

When one wishes to cover an existing position on a given asset by using a forward contract, one ensures that the corresponding asset is as close as possible to the original one. In the case where these two assets are not perfectly correlated, we will see how to compute the optimal quantity of forward contracts to detain.

Let then t be the current date. An investor wishes to hedge a portfolio of m assets with price C_s at $s \geq t$, with maturity T. He decides to hedge by buying, at t, x forward contracts at the price f_t.

1) What is the net value V_T of the portfolio of the investor at T?

2) We denote $\sigma_V^2 := \mathbb{V}\mathrm{ar}(V_T),\ \sigma_C^2 := \mathbb{V}\mathrm{ar}(C_T),\ \sigma_F^2 := \mathbb{V}\mathrm{ar}(f_T),\ \rho_{CF} := \mathbb{C}\mathrm{ov}(C_T, f_T)$. Show that in order to minimise σ_V^2, he must choose

$$x = -m\frac{\rho_{CF}}{\sigma_F^2}.$$

3) Show that the risk can only be canceled if $\rho_{CF}^2 = \sigma_C^2\sigma_F^2$. What does it imply for the assets C and F?

4) Application: an airline knows that that it will need to buy 2M litres of oil in one month. As there are no futures contracts on oil, the airline will use futures on gasoil. We give below the variations[1] of oil price ΔS and the variations of the corresponding futures prices Δf

Month	Δf	ΔS
1	0.021	0.029
2	0.035	0.020
3	-0.046	-0.044
4	0.001	0.008
5	0.044	0.026
6	-0.029	-0.019
7	-0.026	-0.010
8	-0.029	-0.007
9	0.048	0.043
10	-0.006	0.011
11	-0.036	-0.036
12	-0.011	-0.018
13	0.019	0.009
14	-0.027	-0.032
15	0.029	0.023

Given that each futures contract on gasoil is for a quantity of 42000 litres, how many contracts must the company buy? What is its risk?

Swaps

We consider two firms A and B, which can borrow on the market at the following conditions

- Firm A : 8.5% fixed rate or Euribor +3%, for borrowing 10M for 8 years.
- Firm B : 4.5% fixed rate or Euribor +1%, for borrowing 10M for 8 years.

[1]This means here that ΔS is the absolute variation of S between two consecutive months, and similarly for Δf.

They both decide to go to the same bank, which will be in charge to design a swap contract between them (and which thus will take a fee). Depending on the preferences of the firms (fixed rate for A and floating for B, or fixed rate for B and floating for A), find, if they exist, all the swap contracts (including the fees for the bank) which can improve the borrowing conditions of both firms. What is the maximal fee that the bank can get?

Speed of a Bond

We consider a Bond with maturity t_n delivering cash-flows $F(t_i)$ at the successive dates $(t_i)_{1\leq i\leq n}$. We also assume that the interest rate curve is flat at the fixed level $r > 0$. Interest rates are supposed to be compounded once per year.

1) What is the formula giving the price of this Bond at time $t = 0$?

2) Recall the definition of the Duration D and the Convexity C for a Bond, and provide first and second order approximations of the price variation ΔP due to an interest rate variation of Δr.

3) We know define the Speed κ of a Bond as follows

$$\kappa := -\frac{1}{P}\frac{\partial^3 P}{\partial r^3}.$$

Give an explicit formula for κ in this exercise, and its sign. Deduce a third order approximation of the price variation ΔP due to an interest rate variation of Δr, in terms of D, C and κ. Deduce whether we over or underestimate the variations of P when we neglect the third order term.

3) We consider a Bond with maturity 2 years, face value of \$100, a coupon rate of 10%, and we assume $r = 8\%$ and that coupons are paid annually. Compute the value, the Duration, the Convexity and the Speed of this Bond. Compute its new exact price when r diminishes brutally to 1%, as well as the approximated new prices using the first, second, and third order approximations. Comment.