Introduction to Mathematical Finance
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Assignment 5

1. **On American Calls with dividends**

   Let S be a stock paying a known dividend amount κ at some known time $\tau \in (0,T)$.

   1) Prove that the following relationship holds for any $t < \tau$

   $$S_t \geq C_t^A(T,K;S) \geq \max\Big\{\big(S_t - KB(t,\tau)\big)^+, \big(S_t - \kappa B(t,\tau) - KB(t,T)\big)^+\Big\}.$$

   2) Assume now that S pays n known dividends $(\kappa_i)_{1\leq i\leq n}$ at the known dates $(\tau_i)_{1\leq i\leq n}$ with $0<\tau_1<\tau_2<\cdots<\tau_n<T$. Prove that for any $t<\tau_1$

   $$S_t \geq C_t^A(T,K;S) \geq \max_{0\leq i\leq n}\left\{\left(S_t - KB(t,\tau_{i+1}) - \sum_{j=1}^{i}\kappa_j B(t,\tau_j)\right)^+\right\},$$

   with the convention that $\tau_{n+1}=T$ and that a sum over an empty set is 0.

   3) Prove that the only dates where it can be optimal to exercise a Call option on an underlying asset paying n known dividends $(\kappa_i)_{1\leq i\leq n}$ at the known dates $(\tau_i)_{1\leq i\leq n}$ are the maturity T, or at the times $(\tau_i)_{1\leq i\leq n}$, just before the dividends are paid. Under which condition(s) is it not optimal to exercise the American Call prior to T?

2. **Asymptotics for the CRR model**

   We consider a variation on the multi–period binomial model from Section 2.3.2.2. We let our time horizon be some time $T>0$, and take $\Omega := \{\omega^u,\omega^d\}^m$, where m is a positive integer representing the number of periods in the market. As usual, we fix $\mathcal{F} := \mathcal{P}(\Omega)$. We depart a little bit from the lecture notes' notations, and consider that the market trading dates are given by $(t_k^m)_{k\in\{0,\dots,m\}}$, where

   $$t_k^m := \frac{kT}{m},\ k\in\{0,\dots,m\}.$$

   The probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ is again given by

   $$\mathbb{P}[\{\omega\}] = p^{U(\omega)}(1-p)^{m-U(\omega)},\ \forall \omega := (\omega_1,\dots,\omega_m)\in\Omega,$$

   where $p\in(0,1)$, and where for any $\omega\in\Omega$, $U(\omega)$ counts the number of elements of ω which are equal to ω^u.

   The non–risky asset values are given by

   $$S^{m,0}_{t_k^m}(\omega) := \left(1+\frac{rT}{m}\right)^k,\ k\in\{0,\dots,m\},\ \omega\in\Omega,$$

   where $r\in\mathbb{R}$, while that of the risky asset are given, for any $\omega\in\Omega$, by

   $$S^m_{t_0^m}(\omega) := S_0,$$

   $$S^m_{t_{k+1}^m}(\omega) = S^m_{t_{k+1}^m}(\omega_1,\dots,\omega_{k+1}) := \begin{cases}(1+h_m)S^m_{t_k^m}(\omega_1,\dots,\omega_k),\ \text{if } \omega_{k+1}=\omega^u,\\ (1+\ell_m)S^m_{t_k^m}(\omega_1,\dots,\omega_k),\ \text{if } \omega_{k+1}=\omega^d,\end{cases}\quad k\in\{0,\dots,m-1\},$$

   where

   $$\ell_m := \frac{rT}{m} - \sigma_-\sqrt{\frac{T}{m}},\ h_m := \frac{rT}{m} + \sigma_+\sqrt{\frac{T}{m}},$$

for some given $(\sigma_-, \sigma_+) \in (0, +\infty)^2$. Notice that we index the asset values by $m \in \mathbb{N} \setminus \{0\}$ since the aim of the exercise is to let m go to $+\infty$.

We will assume throughout that m is large enough so that

$$\ell_m > -1, \ \frac{rT}{m} > -1.$$

1) Prove that for m large enough this market admits no–arbitrage opportunities, and that there is a unique risk–neutral measure $\mathbb{Q}^m$. We let $\underline{m}$ be the lowest value of m such that this holds.

2) We define

$$p_m := \mathbb{Q}^m \left[\left\{ \frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}} = 1 + h_m \right\} \right], \ k \in \{0, \dots, m-1\}.$$

Give the exact value of p_m and justify that it indeed does not depend on $k \in \{0, \dots, m-1\}$.

3) Which property does the sequence of random variables $\left(\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}} \right)_{k \in \{0,\dots,m-1\}}$ satisfy under $\mathbb{Q}^m$.

4) We denote by Φ_m the characteristic function under $\mathbb{Q}^m$ of the random variable $\log\left(S^m_{t^m_1}/S^m_{t^m_0}\right)$, that is to say

$$\Phi_m(\lambda) := \mathbb{E}^{\mathbb{Q}^m}\left[\exp\left(i\lambda \log\left(\frac{S^m_{t^m_1}}{S^m_{t^m_0}} \right) \right) \right].$$

Show that the following Taylor expansion holds

$$\Phi_m(\lambda) = 1 + \left(i\lambda \left(r - \frac{\sigma_-\sigma_+}{2} \right) - \lambda^2 \frac{\sigma_-\sigma_+}{2} \right) \frac{T}{m} + \circ\left(\frac{1}{m} \right).$$

5) Let $Y_m := \log(S^m_T/S_0)$. Express the characteristic function of Y_m under $\mathbb{Q}^m$ in terms of Φ_m, and then show that the sequence $(Y_m)_{m \geq \underline{m}}$ converges in law, under $\mathbb{Q}^m$, to a random variable whose distribution you'll give explicitly.

Hint: It would be useful here to prove the following lemma.

Lemma 0.1. *For any $(z, R) \in \mathbb{C} \times (0, +\infty)$, if $|z| \leq R$, then for any positive integer n*

$$\left| \mathrm{e}^z - \left(1 + \frac{z}{n} \right)^n \right| \leq \mathrm{e}^R - \left(1 + \frac{R}{n} \right)^n.$$

In addition, for any complex–valued sequence $(z_n)_{n \in \mathbb{N} \setminus \{0\}}$ converging to some $z \in \mathbb{C}$, we have

$$\lim_{n \to +\infty} \left(1 + \frac{z_n}{n} \right)^n = \mathrm{e}^z.$$

6) Prove that for any $K \geq 0$

$$\lim_{m \to +\infty} P_0(T, K; S^m) = \mathrm{e}^{-rT} \int_{\mathbb{R}} \left(K - S_0 \mathrm{e}^{a+bx} \right)^+ \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \mathrm{d}x,$$

where you will explicit the constants a and b in terms of r, σ_-, σ_+, and T.

7) Deduce that there exist constants d_0 and d_1 (which you will again provide explicitly) such that

$$\lim_{m \to +\infty} P_0(T, K; S^m) = \mathrm{e}^{-rT} K \mathcal{N}(-d_1) - S_0 \mathcal{N}(-d_0),$$

where $\mathcal{N}(x) := \int_{-\infty}^x \frac{\mathrm{e}^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \mathrm{d}u$, $x \in \mathbb{R}$ is the repartition function of a standard Gaussian random variable.

Prove that a similar formula holds for $\lim_{m \to +\infty} C_0(T, K; S^m)$.

8) (Optional question) Redo the whole exercise until question 5)

$$\ell_m := \frac{rT}{m} - \sigma_- \frac{T}{m},\ h_m := \frac{rT}{m} + \sigma_+ - \sigma_- \frac{T}{m}.$$

(Notice that obviously the expansion in 4) will now be different.)