Introduction to Mathematical Finance
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Assignment 6

On markets with fixed transaction costs

We consider a one period market with a risk–less asset with return $R \geq 0$. There is only one risky asset. At time 0, it is worth $S_0 \in (0, +\infty)$ and can take two values at time 1, either uS_0 or dS_0, each with probability $1/2$, where $u > d > 0$ are given. A strategy consists in buying Δ shares of the risky asset at 0. The position is cleared at time 1, meaning that if $\Delta \geq 0$, this is a buy at 0 and therefore a sell at 1. Conversely if $\Delta \leq 0$. Each transaction, buy or sell, is subject to a fixed fee $c > 0$, that is to say that each time we buy or sell, we have to pay immediately c, whatever the amount of the transaction is.

We denote by $X_1^{x,\Delta}$ the value at time 1, after having cleared the position, given that we use the strategy Δ and that the initial endowment in cash is x, before the first transaction (no initial position in the risky asset is assumed before the first transaction at 0).

In this model, we say that there is no arbitrage opportunities if we cannot find $\Delta \in \mathbb{R}$ such that

$$\mathbb{P}[X_1^{0,\Delta} \geq 0] = 1, \text{ and } \mathbb{P}[X_1^{0,\Delta} > 0] > 0.$$

1) Prove that for any $(x, \Delta) \in \mathbb{R}^2$

$$X_1^{x,\Delta} = \Delta S_1 + \big(x - \Delta S_0 - c\mathbf{1}_{\{\Delta \neq 0\}}\big)R - c\mathbf{1}_{\{\Delta \neq 0\}}.$$

2) Show that there is no arbitrage opportunities if $d \leq R \leq u$.

3) We now want to prove the converse result, namely that if there are no arbitrage opportunities, then $d \leq R \leq u$.

   (a) Assume that $R < d$. Show that we can find $\Delta > 0$ such that

$$d - R > \frac{(1+R)c}{\Delta S_0}.$$

   Deduce an arbitrage.

   (b) Assume that $R > u$. Show that we can find $\Delta < 0$ such that

$$S_0 \Delta (u - R) > (1+R)c.$$

   Deduce an arbitrage.

4) Provide a necessary and sufficient condition on u, d and R for the absence of arbitrage opportunities.

   From now on, we assume that $R \in [d, u]$.

5) We are now looking for a probability $\mathbb{Q}$ such that for all $\Delta \in \mathbb{R}$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X_1^{0,\Delta}}{R}\right] \leq 0.$$

   (a) By using the notation $q := \mathbb{Q}[S_1 = uS_0] = 1 - \mathbb{Q}[S_1 = dS_0]$, write down the expression of $\mathbb{E}^{\mathbb{Q}}[X_1^{0,\Delta}]$.

   (b) Deduce that for all $\Delta \in \mathbb{R}$

$$\Delta S_0 (qu + (1-q)d - R) \leq (1+R)c\mathbf{1}_{\{\Delta \neq 0\}}.$$

(c) Deduce that

$$qu + (1-q)d = R.$$

(d) Deduce that there exists a unique probability measure $\mathbb{Q}$ such that for all $\Delta \in \mathbb{R}$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X_1^{0,\Delta}}{R}\right] \leq 0,$$

and give its explicit expression.

(e) Under which condition on u, d and R is this measure equivalent to $\mathbb{P}$?

6) Let us consider an European option with payoff ξ defined by $\xi = \xi(u) \in \mathbb{R}$, if $S_1 = uS_0$, and $\xi = \xi(d) \in \mathbb{R}$, if $S_1 = dS_0$. We assume that $\xi(u) \neq \xi(d)$.

(a) Define

$$p^+ := \mathbb{E}^{\mathbb{Q}}\left[\frac{\xi}{R}\right] + c\left(1 + \frac{1}{R}\right).$$

Show that there exists $\Delta \in \mathbb{R}$ such that

$$\mathbb{P}\big[X_1^{p^+,\Delta} = \xi\big] = 1.$$

(b) Define

$$p^- := \mathbb{E}^{\mathbb{Q}}\left[-\frac{\xi}{R}\right] + c\left(1 + \frac{1}{R}\right).$$

Show that there exists $\Delta \in \mathbb{R}$ such that

$$\mathbb{P}\big[X_1^{p^-,\Delta} = -\xi\big] = 1.$$

(c) What is the set of viable prices for ξ (that is to say the set of prices for ξ which do not create arbitrage opportunities in the market, see the lecture notes for details) in this model?

On markets with price impact

We consider a one period market with a risk–less asset with return $R = 1$. There is only one risky asset, and the space Ω will simply be the pair $\{\omega^1, \omega^2\}$.

We assume that a large trader invests in this market and has an impact on prices an generates a liquidity cost. The reference price at time 0 for the risky asset, before the initial trade, is $S_0 > 0$. When buying $\Delta_0 > 0$ shares at time 0, the trader has to pay

$$\Delta_0(S_0 + \lambda\Delta_0),$$

where $\lambda > 0$ is a given constant.

Similarly, the gain that the trader makes from a sell of $\Delta_0 < 0$ at time 0 is

$$-\Delta_0(S_0 + \lambda\Delta_0).$$

The reference price for the risky asset at time 1 depends on the impact of Δ_0 at time 0. More precisely

$$S_1^{\Delta_0}(\omega^i) := u_i\left(S_0 + \lambda\frac{\Delta_0}{2}\right),\ i = 1, 2,$$

where as usual $u_1 > u_2 > 0$.

Again, at time 1, the cost of buying $\Delta_1 \geq 0$ shares is

$$\Delta_1\big(S_1^{\Delta_0} + \lambda\Delta_1\big),$$

and the gain that the trader makes from a sell of $\Delta_1 < 0$ is

$$-\Delta_1\big(S_1^{\Delta_0} + \lambda\Delta_1\big).$$

Since we will only consider trading strategies where the portfolio of the investor is liquidated at time 1, any transaction of $\Delta \in \mathbb{R}$ assets at time 0 will be associated to a transaction of $-\Delta$ assets at time 1. We denote the corresponding portfolio values as $(X_t^{x,\Delta})_{t=0,1}$, $x \in \mathbb{R}$ being the initial capital available to the trader.

1) Consider the strategy $\Delta \in \mathbb{R}$ with 0 initial capital. Show that the corresponding gain is, for $i = i, 2$

$$X_1^{0,\Delta}(\omega^i) = \Delta(u_i - 1)S_0 + \frac{\Delta^2}{2}\lambda(u_i - 4).$$

   In the following, we will say that Condition (**NA**) holds if

$$\text{There is no } \Delta \in \mathbb{R}, \text{ s.t. } \mathbb{P}[X_1^{0,\Delta} \geq 0] = 1,\ \mathbb{P}[X_1^{0,\Delta} > 0] > 0.$$

2) Show that we can find an arbitrage by choosing $|\Delta|$ small, when $u_2 > 1$, or when $u_1 < 1$.

3) Let us assume in this question that $u_2 \leq 1 \leq u_1 \leq 4$, and fix some $\Delta \in \mathbb{R}$. Show that $X_1^{0,\Delta}(\omega^2) \geq 0$ implies that necessarily $\Delta \leq 0$. Show as well that $\Delta < 0$ implies that $X_1^{0,\Delta}(\omega^1) < 0$. Deduce that $u_2 \leq 1 \leq u_1 \leq 4$ implies Condition (**NA**).

4) Deduce a necessary and sufficient condition on u_1 and u_2 which ensures that Condition (**NA**) holds in the case $u_1 \leq 4$. Does it imply the existence of a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ such that

$$\mathbb{E}^{\mathbb{Q}}[S_1^0] = S_0?$$

   And what if we consider S_1^{Δ} for an arbitrary $\Delta \in \mathbb{R}$, instead of S_1^0? Comment.

5) Let us assume in this question that $u_2 \leq 1 \leq u_1$, and $u_1 > 4$.

   (a) Show that $\mathbb{P}[X_1^{0,\Delta} \geq 0] = 1$ implies that $\Delta \leq 0$ and

$$m := \frac{2(u_1 - 1)S_0}{\lambda(u_1 - 4)} \leq |\Delta| \leq \frac{2|u_2 - 1|S_0}{\lambda|u_2 - 4|} =: M.$$

   (b) Show that one can construct an arbitrage if $m < M$. Is this inequality possibly satisfied?

   (c) What does it imply in terms of price manipulation?