Introduction to Mathematical Finance
Dylan Possamaï

Assignment 7

About hedging

We consider a T-period binomial market with a risk-less asset with constant return $R > 0$. This means in particular that

$$S_t^0 = R^t,\ t \in \{0, \dots, T\}.$$

There is only one risky asset. At time 0, it is worth $S_0 \in (0, +\infty)$ and there is $0 < d < u$ such that

$$S_{t+1}(\omega) = \big(\mathbf{1}_{\{\omega=\omega^u\}}u + \mathbf{1}_{\{\omega=\omega^d\}}d\big)S_t,\ t \in \{0, \dots, T-1\}.$$

1)a) Recall the condition ensuring that (**NA**) holds in this market.

1)b) Let us be given a European option with maturity T, with payoff $h(S_T)$ for some map $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, and we denote by p_t the price process for this option (that is $p_t(\omega)$ is the value of this option at time t when the realisation of the world is $\omega \in \Omega$. Prove that it is possible to find a map $v : \{0, \dots, T\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$p_t(\omega) = v\big(t, S_t(\omega)\big),\ (t, \omega) \in \{0, \dots, T\} \times \Omega.$$

In particular, you will give a recursive procedure allowing to compute v.

1)c) Let $(x, \Delta) \in \mathbb{R}_+ \times \mathcal{A}(\mathbb{R})$ be a replication strategy for the aforementioned European option. Show that you can find a map $\varphi : \{0, \dots, T-1\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\Delta_t(\omega) = \varphi\big(t, S_t(\omega)\big),\ (t, \omega) \in \{0, \dots, T-1\} \times \Omega.$$

1)d) Show that if the map $x \longmapsto h(x)$ is monotone, then for any $t \in \{0, \dots, T\}$, the map $x \longmapsto v(t, x)$ has the same monotony. Deduce that that whenever $x \longmapsto h(x)$ is non-decreasing, $\varphi \geq 0$, and whenever $x \longmapsto h(x)$ is non-increasing, then $\varphi \leq 0$. How can you interpret this result?

2) We suppose throughout this question that $x \longmapsto h(x)$ is convex.

2)a) Show that for any $t \in \{0, \dots, T\}$, the map $x \longmapsto v(t, x)$ is also convex.

2)b) Show that for any $(x, y, z) \in \mathbb{R}_+^3$ such that $x < y < z$, we have

$$\frac{h(y) - h(x)}{y - x} \leq \frac{h(z) - h(x)}{z - x} \leq \frac{h(z) - h(y)}{z - y}.$$

2)c) Deduce that the following two quantities are well-defined (notice that we allow them here to take the value $+\infty$)

$$L := \lim_{x \to +\infty} \frac{h(x)}{x}, \text{ and } \ell := \lim_{x \to 0} \frac{h(x) - h(0)}{x},$$

and then that for any $0 \leq x < y$

$$\ell \leq \frac{h(y) - h(x)}{y - x} \leq L.$$

2)d) Show that for any $t \in \{0, \dots, T\}$ the map $x \longmapsto v(t, x)$ satisfies the same inequalities as h in 2)c), and then that

$$\ell \leq \varphi(t, x) \leq L,\ (t, x) \in \{0, \dots, T-1\} \times \mathbb{R}_+.$$

What can you deduce for European Call and Put options?

3)a) Let us define for any $0 \leq a \leq A \leq +\infty$ the set

$$\mathcal{E}_{a,A} := \left\{ w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \forall (x,y)\mathbb{R}_+^2,\ x \neq y,\ \text{we have } a \leq \frac{w(y)-w(x)}{y-x} \leq A \right\}.$$

Show that for any $\lambda \in [0,1]$, and for any $(\alpha,\beta) \in (0,+\infty)^2$, the transformation $\Theta_{\lambda,\alpha,\beta}$ defined on $\mathcal{E}_{a,A}$ by

$$\Theta_{\lambda,\alpha,\beta}(w)(x) := \frac{\lambda w(\alpha x) + (1-\lambda) w(\beta x)}{\lambda \alpha + (1-\lambda)\beta},\ x \in \mathbb{R}_+,\ w \in \mathcal{E}_{a,A},$$

is an homomorphism (that is to say that the codomain of $\Theta_{\lambda,\alpha,\beta}$ is $\mathcal{E}_{a,A}$).

3)b) Deduce that if $h \in \mathcal{E}_{a,A}$, then

$$a \leq \varphi(t,x) \leq A,\ (t,x) \in \{0,\dots,T-1\} \times \mathbb{R}_+.$$

3)c) Show, using an example, that the result of 3)c) for the replicating strategy is more general than the result of 2)c).

4)a) We now consider an American option with maturity T and payoff $h(S_t)$ when it is exercised at time $t \in \{0,\dots,T\}$. You will admit that if p_t is the value of this option at time t, then p_t satisfies the following backward induction (where $\mathbb{Q}$ is the only risk-neutral measure on the market)

$$p_T(\omega) = h\big(S_T(\omega)\big),\ p_t(\omega) = \max\left\{ h\big(S_t(\omega)\big), \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[p_{t+1}|\mathcal{F}_t](\omega) \right\},\ (t,\omega) \in \{0,\dots,T-1\} \times \Omega.$$

How can you interpret this formula?

You will also admit that the replicating strategy for such an American option can be obtained, *mutatis mutandis*, with the same recursive formula as for European options. Deduce then that, as in the European option case, we can find a map $v^a : \{0,\dots,T\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$p_t(\omega) = v^a\big(t, S_t(\omega)\big),\ (t,\omega) \in \{0,\dots,T\} \times \Omega,$$

and if $(x,\Delta) \in \mathbb{R}_+ \times \mathcal{A}(\mathbb{R})$ is a replicating strategy for the American option, we can find a map $\varphi^a : \{0,\dots,T-1\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\Delta_t(\omega) = \varphi^a\big(t, S_t(\omega)\big),\ (t,\omega) \in \{0,\dots,T-1\} \times \Omega.$$

4)b) Answer once more to questions 1)d) and 2)a) in this context.

4)c) Assume that h is convex, and prove, with the same notations as in 2), that

$$\big|\varphi^a(t,x)\big| \leq \max\big\{|\ell|, |L|\big\},\ (t,x) \in \{0,\dots,T-1\} \times \mathbb{R}_+.$$

4)d) Show that a similar result holds when h is Lipschitz-continuous.

4)e) Assume now that $h \in \mathcal{E}_{a,A}$. Does the result of 3)b) extend to the current context?

4)f) (Optional). Explain how you would extend the results of 4)a)–4)e) to an American option whose payoff at time $t \in \{0,\dots,T\}$ is now of the form $h(t,S_t)$, for some map $h : \{0,\dots,T\} \times \mathbb{R}_+$.

Sharpness of call options bounds

The goal of this exercise is to exhibit a financial market in which the bounds

$$(S_t - KB(t,T))^+ \leq C_t(T,K;S) \leq S_t, \tag{0.1}$$

are attained. We thus fix a measurable space $(\Omega, \mathcal{F})$ defined as follows: $\Omega := (0, +\infty)$, and $\mathcal{F}$ is the Borel–σ-algebra on Ω. We let X be the canonical map on Ω, that is

$$X(\omega) = \omega, \ \omega \in \Omega,$$

and we take a probability $\mathbb{P}$ measure on $(\Omega, \mathcal{F})$ making X into a standard log-normal random variable (that is $\log(X)$ has a standard Gaussian distribution).

The model has $T = d = 1$, and we take $\mathcal{F}_0$ trivial, as well as $\mathcal{F}_1 := \mathcal{F}$. The asset prices are given, for some $r \geq 0$, by

$$S_0^0 = 1, \ S_1^0 = \mathrm{e}^r, \ S_0 = 1, \ S_1 = X.$$

1)a) Show that $\mathcal{F}_1 = \mathcal{F} = \sigma(X) = \sigma(S_1)$.

1)b) Show that the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ with density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \exp\left(\left(r - \frac{1}{2}\right)\log(X) - \frac{1}{2}\left(r - \frac{1}{2}\right)^2\right),$$

is well-defined and is a risk-neutral measure.

1)c) Show that the market is however incomplete by constructing a non-replicable payoff.

2) Let $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$. We now define a subset $\mathcal{P}_{\text{bin}}$ of $\mathcal{P}$, consisting of all martingale measures for S which in addition make the market into a binomial one, that is to say[1]

$$\mathcal{P}_{\text{bin}} := \Big\{\Pi \in \mathcal{P} : \Pi \circ (S)^{-1} \text{ has mass in two points, and } \mathbb{E}^{\Pi}\big[\mathrm{e}^{-r} S_1\big] = 1\Big\}.$$

2)a) Are elements of $\mathcal{P}_{\text{bin}}$ risk-neutral measures? Why?

2)b) Fix some $0 < d < \mathrm{e}^r < u$. Construct a sequence $(\Pi_n)_{n \in \mathbb{N}}$ of measures in $\mathcal{M}(S)$ (which thus must be equivalent to $\mathbb{P}$), but which converges weakly to some $\Pi \in \mathcal{P}_{\text{bin}}$ which has mass at u and d.

2)c) Define now the set

$$\mathfrak{P}_{\text{bin}} := \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1 - K)^+\big] : \Pi \in \mathcal{P}_{\text{bin}}\Big\}.$$

Show that

$$\mathfrak{P}_{\text{bin}} \subset \Big[-p\big(-(S_1 - K)^+\big), p\big((S_1 - K)^+\big)\Big].$$

Hint: it could be useful to use convex combinations of $\mathbb{Q}$ *and elements of the sequences* $(\Pi_n)_{n \in \mathbb{N}}$ *from* 2)b).

2)d) Show that

$$\sup_{\Pi \in \mathcal{P}_{\text{bin}}} \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1 - K)^+\big]\Big\} = 1, \ \inf_{\Pi \in \mathcal{P}_{\text{bin}}} \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1 - K)^+\big]\Big\} = (1 - K)^+,$$

and deduce that the universal bounds in (0.1) (for $t = 0$) are attained in this market.

[1]The notation $\Pi \circ (S)^{-1}$ represents the distribution of S under Π. In more measure-theoretic terms, this is simply the image measure of Π through the measurable map $S : \Omega \longrightarrow (0, +\infty)$.