Introduction to Mathematical Finance
Dylan Possamaï

Assignment 8

About Asian options

We consider a complete T-period financial market, such that (**NA**) holds. There is a risk-less asset which is for now such that $(1/S_t^0)_{t\in\{0,\dots,T\}}$ is a positive $(\mathbb{F},\mathbb{Q})$-super-martingale, where $\mathbb{Q}$ is the unique risk-neutral measure on this market. There is only one risky asset with price process S.

We fix some $K \geq 0$, and we are interested in a so-called Asian Call option on S, whose payoff at maturity T is given by

$$\left(\frac{1}{T}\sum_{k=1}^{T} S_k - K\right)^+.$$

We will denote by $C_t^{\mathrm{as}}(T,K;S)$ the value at any time $t\in\{0,\dots,T\}$ of such an option. For notational simplicity, we will also take the convention in the formulae below that $\frac{0}{0}=0$.

1)a) Show that $\mathbb{P}$-a.s.

$$\left(\frac{1}{T}\sum_{k=1}^{T} S_k - K\right)^+ \leq \frac{1}{T}\sum_{k=1}^{T}(S_k-K)^+.$$

1)b) Deduce that

$$C_0^{\mathrm{as}}(T,K;S) \leq \frac{1}{T}\sum_{k=1}^{T} C_0(k,K;S).$$

2)a) Show that for any $t\in\{0,\dots,T\}$ and any $s\in\{t,\dots,T\}$

$$\frac{(S_t-K)^+}{S_t^0} \leq \left(\widetilde{S}_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{K}{S_s^0}\bigg|\mathcal{F}_t\right]\right)^+,\ \mathbb{P}\text{-a.s.}$$

2)b) Deduce using Jensen's inequality for conditional expectations that for any $t\in\{0,\dots,T\}$, with $\mathbb{P}$-probability one, the sequence $(C_t(k,K;S))_{k\in\{s,\dots,T\}}$ is non-decreasing.

2)c) Show that

$$C_0^{\mathrm{as}}(T,K;S) \leq C_0(T,K;S).$$

3) In this question we will extend the previous results to any time $t\in\{0,\dots,T\}$.

3)a) Show that for any $t\in\{0,\dots,T\}$, we have $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T,K;S) \leq \frac{t}{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\bigg|\mathcal{F}_t\right]\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T} C_t(k,K;S).$$

3)b) Show that the result in 3)a) is indeed a generalisation of 2)c).

4) From now on, and until the end of the problem, we assume that S^0 is deterministic. Prove that we can now write for any $t\in\{0,\dots,T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T,K;S) \leq \frac{t}{T}\frac{S_t^0}{S_T^0}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}C_t(k,K;S).$$

5) Define for any $t \in \{0, \dots, T\}$ the event

$$A(t) := \left\{ \frac{1}{T} \sum_{k=1}^{t} S_k \geq K \right\} \in \mathcal{F}_t.$$

Show that for any $t \in \{0, \dots, T\}$, $\mathbb{P}$-a.s.

$$\mathbf{1}_{A(t)} C_t^{\mathrm{as}}(T, K; S) = \mathbf{1}_{A(t)} \left(\frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{t} S_k - K \right) + \frac{S_t}{T} \sum_{k=t+1}^{T} \frac{S_k^0}{S_T^0} \right).$$

6) Fix now some $T_o \in \{1, \dots, T-1\}$ and consider the option with maturity T and payoff

$$\left(\frac{1}{T - T_o} \sum_{k=T_o+1}^{T} S_k - K \right)^+.$$

We denote by $C^{\mathrm{as}}(T_o, T, K; S)$ the price process of the corresponding option.

6)a) Show that for any $t \in \{T_o, \dots, T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T_o, T, K; S) \leq \frac{t - T_o}{T - T_o} \frac{S_t^0}{S_T^0} \left(\frac{1}{t - T_o} \sum_{k=T_o+1}^{t} S_k - K \right)^+ + \frac{1}{T - T_o} \sum_{k=t+1}^{T} \frac{S_k^0}{S_T^0} C_t(k, K; S).$$

6)b) Show that for any $t \in \{T_o, \dots, T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T_o, T, K; S) \leq \frac{1}{T - T_o} \sum_{k=T_o+1}^{T} \frac{S_k^0}{S_T^0} C_t(k, K; S).$$

7) We now specialise the discussion to a more specific model and assume that for some $R > 0$

$$S_t^0 = R^t, \ t \in \{0, \dots, T\},$$

and that the risky asset satisfies that $S_0 > 0$ is given and

$$S_{t+1} = Y_{t+1} S_t, \ t \in \{0, \dots, T-1\},$$

where the $(Y_i)_{i \in \{1, \dots, T\}}$ are i.i.d. random variables under $\mathbb{Q}$, taking values in $(0, +\infty)$.

7)a) Explain first why

$$\mathbb{E}^{\mathbb{Q}}[Y_i] = R, \ i \in \{1, \dots, T\}.$$

7)b) Prove then that, $\mathbb{P}$-a.s.

$$\left(\frac{1}{T} \sum_{k=1}^{T} S_k - K \right)^+ \geq \left(S_0 \exp\left(\sum_{k=1}^{T} \left(1 - \frac{k-1}{T} \right) \log(Y_k) \right) - K \right)^+,$$

and then that

$$C_0^{\mathrm{as}}(T, K; S) \geq \frac{1}{R^T} \mathbb{E}^{\mathbb{Q}} \left[\left(S_0 \exp\left(\sum_{k=1}^{T} \frac{k}{T} \log(Y_k) \right) - K \right)^+ \right].$$

7)c) Define the following process

$$\overline{S}_t := S_0 \exp\left(\frac{1}{t} \sum_{k=1}^{t} k \log(Y_k) \right), \ t \in \{0, \dots, T\}.$$

Show that the lower bound obtained in 7)b) can be written formally

$$C_0^{\mathrm{as}}(T, K; S) \geq C_0(T, K; \overline{S}).$$

Can we however say that $\overline{S}$ is the price process of a (fictitious) risky asset in this market?