Introduction to Mathematical Finance
Dylan Possamaï

Assignment 9

About Futures

We consider a complete T-period financial market, such that (**NA**) holds, and we let $\mathbb{Q}$ be the unique risk-neutral measure on this market.

Futures contracts, unlike forward contracts, are marked-to-market, meaning that they receive cash-flows at every trading dates. More precisely, a futures contract is an agreement to purchase an asset at the maturity T, for a pre-specified price, called the *futures price*. This futures price is paid via a sequence of instalments over the contract's life. As with forward contracts, no cash-flow happens at the inception of the contract, supposed to correspond to time 0 here. However, a cash payment is made at every trading date, corresponding to the change in the futures price between this date and the previous trading one. Mathematically, if we define the futures price at time t, for an asset S with maturity T by $G_t(T;S_T)$, then the cash-flows are

$$G_t(T;S_T) - G_{t-1}(T;S_T), \text{ at time } t \in \{1,\dots,T\}.$$

Explain why the value V_t of a futures contract is 0 at any time $t \in \{0,\dots,T\}$. Show as well that

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=t+1}^{T} d(t,k)\big(G_k(T;S_T) - G_{k-1}(T;S_T)\big)\bigg|\mathcal{F}_t\right].$$

Prove then that $G_T(T;S_T) = S_T$ and deduce from all the above that the futures prices are actually given by

$$G_t(T;S_T) = \mathbb{E}^{\mathbb{Q}}\big[S_T\big|\mathcal{F}_t\big],\ t \in \{0,\dots,T\}.$$

Show then that the difference between forward and futures prices is given by

$$F_t(T;S_T) - G_t(T;S_T) = \frac{\mathbb{C}\mathrm{ov}^{\mathbb{Q}}[S_T, d(t,T)|\mathcal{F}_t]}{B(t,T)},\ t \in \{0,\dots,T\}.$$

Which of $F_t(T;S_T)$ or $G_t(T;S_T)$ would you expect is usually the largest? Why?

Numéraire change and applications

We fix a general complete and arbitrage-free financial market in discrete-time with horizon $T \in \mathbb{N}\setminus\{0\}$, as described in the lecture notes. We let $\mathbb{Q}$ be the unique risk-neutral on this market, and to avoid any issues with integrability requirements, we assume that all assets appearing have bounded prices.

1) Instead of using the risk-less asset S^0 as numéraire, the goal of this question is to examine what happens if we use another asset. Without loss of generality, we will thus the first risky asset S^1. Let ξ be the payoff at time T of an option, and let $(p_t(\xi))_{t\in\{0,\dots,T\}}$ be the associated no-arbitrage price.

   Define the probability measure $\mathbb{P}^1$ on $(\Omega,\mathcal{F})$ by

$$\frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{Q}} := \frac{S^1_T}{S^1_0 S^0_T}.$$

1)*a*) Show that $\mathbb{P}^1$ is well-defined.

1)*b*) We define the S^1-discounted value of any process $(V_t)_{t\in\{0,\dots,T\}}$ by

$$V^{S^1}_t := \frac{V_t}{S^1_t},\ t \in \{0,\dots,T\}.$$

   Prove that $\widetilde{V}$ is an $(\mathbb{F},\mathbb{Q})$-martingale if and only if V^{S^1} is an $(\mathbb{F},\mathbb{P}^1)$-martingale.

1)c) Deduce that we can write

$$p_t(\xi) = \mathbb{E}^{\mathbb{P}^1}\left[\frac{S^1_t}{S^1_T}\xi\middle|\mathcal{F}_t\right].$$

This formula shows that the risk-neutral pricing method is invariant under the so-called *change of numéraire*.

2) Caps and floors are to PFS and RFS what call and put options are to forward contracts. In other words, they correspond to IRS contracts where exchange at each payment date only occurs if the payoff is positive. More precisely, we fix a number of payments $n \in \mathbb{N}\setminus\{0\}$, and a sequence $\mathcal{T} := (T_i)_{i\in\{0,\dots,n\}}$ (all belonging to $\{0,\dots,T\}$) of dates, as well as a face–value $N > 0$ and a strike $K > 0$. The discounted payoff at time $t \leq T_0$ of a cap is given by

$$N\sum_{i=1}^n d(t,T_i)(T_i - T_{i-1})\big(\ell(T_{i-1},T_i) - K\big)^+,$$

while that of a floor is

$$N\sum_{i=1}^n d(t,T_i)(T_i - T_{i-1})\big(K - \ell(T_{i-1},T_i)\big)^+.$$

Caps and floors are actually constituted of a stream of simpler contracts, with discounted payoffs of the form

$$Nd(t,T_i)(T_i - T_{i-1})\big(\ell(T_{i-1},T_i) - K\big)^+,\ \text{and}\ Nd(t,T_i)(T_i - T_{i-1})\big(K - \ell(T_{i-1},T_i)\big)^+,$$

which are called respectively the i-th caplet associated to the cap, and the i-th floorlet associated to the floor.

2)a) For any $t \in \{0,\dots,T_0\}$, we let $\mathrm{CAPL}_t(T_{i-1},T_i,N,K)$ be the price at t of the i-th caplet associated to the cap, and $\mathrm{FLOORL}_t(T_{i-1},T_i,N,K)$ the price at t of the i-th floorlet associated to the floor. Show that

$$\begin{aligned}\mathrm{CAPL}_t(T_{i-1},T_i,N,K) &= N(1+(T_i - T_{i-1})K)\mathrm{ZBP}_t\big(T_{i-1},T_i,(1+(T_i-T_{i-1})K)^{-1}\big),\\ \mathrm{FLOORL}_t(T_{i-1},T_i,N,K) &= N(1+(T_i - T_{i-1})K)\mathrm{ZBC}_t(T_{i-1},T_i,(1+(T_i-T_{i-1})K)^{-1}\big),\end{aligned}$$

where for any $0 \leq t \leq k \leq s \leq T$, $\mathrm{ZBC}_t(k,s,L)$ is the value at t of a call option with maturity k, written on a zero-coupon bond with maturity s, and with strike $L \geq 0$, and $\mathrm{ZBP}_t(k,s,L)$ is the value at t of a put option with maturity k, written on a zero-coupon bond with maturity s, and with strike $L \geq 0$.

2)b) Now for any $s \in \{0,\dots,T\}$, we let the s-forward martingale measure (or forward martingale measure with maturity s) correspond to choosing the zero-coupon bond with maturity s as numéraire. This of course means that the measure $\mathbb{P}^s := \mathbb{P}^{B(\cdot,s)}$ is defined on the probability space $(\Omega,\mathcal{F}_s)$ only, since $B(\cdot,s)$ ceases to exist after time s. Show that for any $0 \leq t \leq k \leq s \leq T$

$$\begin{aligned}\mathrm{ZBC}_t(k,s,K) &= B(t,k)\mathbb{E}^{\mathbb{P}^k}\left[\big(F_k(k;B(k,s)) - K\big)^+\middle|\mathcal{F}_t\right],\\ \mathrm{ZBP}_t(k,s,K) &= B(t,k)\mathbb{E}^{\mathbb{P}^k}\left[\big(K - F_k(k;B(k,s))\big)^+\middle|\mathcal{F}_t\right].\end{aligned}$$

2)c) Deduce a formula for the cap and the floor described above in terms of expectations under the forward martingale measures with maturities $(T_i)_{i\in\{0,\dots,n-1\}}$.

2)d) Which advantage do you see in these formulae compared to the ones written under the risk-neutral measure $\mathbb{Q}$?

3) We now consider *swaptions*. These are simply options whose underlying is an IRS. As usual, there are two main types of such options: the payer one (call-like), and the receiver one (put-like). More precisely, a European payer swaption is an option giving the right (and thus not the obligation) to enter a payer IRS at a given future time, which is called the swaption maturity. Usually, this maturity coincides with the first reset date of the underlying IRS. The underlying IRS length, that is to say $T_n - T_0$ with our previous notations, is called the *tenor* of the swaption. It is also commonplace to call *tenor structure* the set of reset and payment dates $\mathcal{T}$. Using T_0 as our maturity date, the payoff at maturity of a payer-swaption, discounted from some time $0 \leq t \leq T_0$ is therefore given by

$$d(t,T_0)\big(\mathrm{PFS}_{T_0}(\mathcal{T},N,K)\big)^+.$$

3)a) Show that

$$\mathrm{PFS}_{T_0}(\mathcal{T},N,K) = N\sum_{i=1}^{n} B(T_0,T_i)(T_i-T_{i-1})\big(\ell(T_0,T_{i-1},T_i)-K\big).$$

3)b) Show that the payoff of a payer-swaption can then be rewritten

$$Nd(t,T_0)\big(s(T_0,\mathcal{T})-K\big)^+\sum_{i=1}^{n}(T_i-T_{i-1})B(T_0,T_i),$$

where the forward–swap rate $s(t,\mathcal{T})$ at time t for the sets of times $\mathcal{T}$ is given by

$$s(t,\mathcal{T}) := \frac{B(t,T_0)-B(t,T_n)}{\sum_{i=1}^{n}(T_i-T_{i-1})B(t,T_i)}.$$

3)c) Let us define the so-called *level process* G by

$$G_t := \sum_{i=1}^{n}(T_i-T_{i-1})B(t,T_i).$$

We let Π correspond to the probability measure defined in 1) when choosing G as a numéraire. Show that the forward–swap rate $s(\cdot,\mathcal{T})$ is an $(\mathbb{F},\Pi)$-martingale, and then that if $\mathrm{PSWAP}_t(\mathcal{T},N,K)$ represents the value at time $0\leq t\leq T_0$ of the payer-swaption, we have

$$\mathrm{PSWAP}_t(\mathcal{T},N,K) = NG_t\mathbb{E}^{\Pi}\big[(s(T_0,\mathcal{T})-K)^+\big|\mathcal{F}_t\big].$$