Introduction to Mathematical Finance
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Assignment 1 – Solutions

Bonds and accrued interest

1) We consider a bond with a face value of \$500, a coupon rate of 5%, emitted on 10.25.N, reimbursed at par value on the 10.25.N+5. If coupon payments are annual, what was the accrued interest on the 12.12.N+3?

   On 12.12.N+3, the last coupon payment happened 10.25.N+3, that is to say 48 days before. Besides, there are 317 days until the next coupon payment. The corresponding accrued interest is thus equal to

$$\textbf{Accrued interest} = \frac{48}{48+317} \times 5\% \approx 0.657\%.$$

2) We consider a bond with a face value of \$1000, quoted today at 65, with accrued interest (in %): 7.396. What is the price of the bond today?

   If the bond is quoted at 65 with accrued interest of 7.396**, its price today is given by**

$$P = 1000 \times 65\% + 1000 \times 7.396\% = \$723.96.$$

3) Suppose that you invest today in a bond with face value of \$1000, a coupon rate of 5%, an emission price of \$995, and a 4 years maturity. We assume that coupon payments are annual. Compute the associated yield to maturity.

   The YTM λ should verify the equation

$$995 = \sum_{i=1}^{4} \frac{1000 \times 5\%}{(1+\lambda)^i} + \frac{1000}{(1+\lambda)^4} = \frac{1000 \times 5\%}{\lambda}\left(1 - \frac{1}{(1+\lambda)^4}\right) + \frac{1000}{(1+\lambda)^4} \Longleftrightarrow \lambda \approx 5.14\%.$$

4) Consider 600 bonds, emitted on 7.1.N, with face value \$1000, reimbursement price \$1010, coupon rate 4%, reimbursed in 5 years. Coupon payments are annual.

   4)*a*) Calculate the yield to maturity of these bonds.

   The YTM λ should verify the equation

$$1000 = \sum_{i=1}^{5} \frac{1000 \times 4\%}{(1+\lambda)^i} + \frac{1010}{(1+\lambda)^5} = \frac{1000 \times 4\%}{\lambda}\left(1 - \frac{1}{(1+\lambda)^5}\right) + \frac{1010}{(1+\lambda)^5} \Longleftrightarrow \lambda \approx 4.18\%.$$

   4)*b*) What is the value of the bond on the 4.5.N+3, given that the market interest rate is 6%. What is then the quoted price of the bond?

   Notice that the next coupon payment will be on 7.1.N+3, that is to say in 87 days. The value of the bond is thus given by

$$P = \sum_{i=0}^{2} \frac{1000 \times 4\%}{(1+6\%)^{i+87/365}} + \frac{1010}{(1+6\%)^{2+87/365}} \approx \$998.27.$$

Since the last coupon payment was 278 days ago, the accrued interest on 4.5.N+3 is

$$\textbf{Accrued interest} = \frac{278}{278+87} \times 4\% \approx 3.05\%.$$

Hence the quoted price is

$$998.27 - 1000 \times 3.05\% = \$967.77.$$

Bonds and YTM

On October 28th 2018, firm X emits bonds with the following characteristics

- Emission price: 990.40
- Face value: $1000
- Coupon rate: 3,75%, coupons are paid annually.
- Maturity: 5 years.
- Reimbursed at par value.

1) What is the yield to maturity on the emission date?

   The YTM λ should verify the equation

$$990.40 = \sum_{i=1}^{5} \frac{1000 \times 3.75\%}{(1+\lambda)^i} + \frac{1000}{(1+\lambda)^5} = \frac{1000 \times 3.75\%}{\lambda}\left(1 - \frac{1}{(1+\lambda)^5}\right) + \frac{1000}{(1+\lambda)^5} \Longleftrightarrow \lambda \approx 3.96\%.$$

2) Assuming that on October 28th 2021 and on October 28th 2022, the market interest rate went from 4.50% to 3%, compute on each of these dates the price of the bond.

   On October 28th 2021, we have

$$P = \sum_{i=1}^{2} \frac{1000 \times 3.75\%}{(1+4.5\%)^i} + \frac{1000}{(1+4.5\%)^2} \approx \$985.95.$$

   On October 28th 2022, we have

$$P = \frac{1000 \times 3.75\%}{1+3\%} + \frac{1000}{1+3\%} \approx \$1007.28.$$

3) How does the market value of the bond change when the market interest rate changes as well? Mention two indicators allowing to measure the exposure of bond to the interest rate risk.

   As expected, the market value of the bond decreases with the interest rate prevailing in the market. The degree of this sensitivity can be measures using the Duration or the Convexity of the bond.

Bonds and compounding

We consider the following term–structure of interest rates

Maturity	ZC rate
1 year	4.00%
2 years	4.50%
3 years	4.75%
4 years	4.90%
5 years	5.00%

1) What is the price of a bond with maturity 5 years, face value \$100 and an annual coupon rate of 5%?

As usual, the value is given by

$$P = \frac{100 \times 5\%}{1 + 4\%} + \frac{100 \times 5\%}{(1 + 4.5\%)^2} + \frac{100 \times 5\%}{(1 + 4.75\%)^3} + \frac{100 \times 5\%}{(1 + 4.9\%)^4} + \frac{105}{(1 + 5\%)^5} \approx \$100.136.$$

2) What is its yield to maturity?

The YTM should verify

$$100.136 = \sum_{i=1}^{5} \frac{100 \times 5\%}{(1 + \lambda)^i} + \frac{100}{(1 + \lambda)^5} \iff \lambda \approx 4.97\%.$$

3) We suppose that the spot curve increases instantaneously and uniformly by 0.5%. What is the new price and the new yield to maturity of the bond? What is the impact of this rates increase for the bondholder?

The new price of the bond becomes

$$P = \frac{100 \times 5\%}{1 + 4.5\%} + \frac{100 \times 5\%}{(1 + 5\%)^2} + \frac{100 \times 5\%}{(1 + 5.25\%)^3} + \frac{100 \times 5\%}{(1 + 5.4\%)^4} + \frac{105}{(1 + 5.5\%)^5} \approx \$97.999.$$

As for the new YTM, we should have

$$97.999 = \sum_{i=1}^{5} \frac{100 \times 5\%}{(1 + \lambda)^i} + \frac{100}{(1 + \lambda)^5} \iff \lambda \approx 5.47\%.$$

The impact of this increase for the bondholder is an absolute capital loss of \$2.137

4) We suppose now that the spot curve will remain stable over time. You hold the bond until maturity. What is the annual return rate of your investment? Why is this rate different from the yield to maturity?

Before maturity, the bondholder receives intermediate coupons of \$5 **that he reinvests on the market, at the prevailing rate. He also receives the terminal payment of \$105 after 5 years. His total income during the operation is thus**

$$5 \times (1 + 4.9\%)^4 + 5 \times (1 + 4.75\%)^3 + 5 \times (1 + 4.5\%)^2 + 5 \times (1 + 4\%) + 105 \approx \$127.461.$$

The corresponding annual return rate is thus

$$\left(\frac{127.461}{100.136}\right)^{1/5} - 1 \approx 4.94\%.$$

This return rate is different from the yield to maturity of this bond (4.97%), because the curve is not flat at this level.

Bootstrap method

We suppose known the following spot rates for maturities less than one year.

Maturity	Spot rate (annual)
1 day	3.20%
1 month	3.30%
2 months	3.40%
3 months	3.50%
6 months	3.60%
9 months	3.80%
1 year	3.20%

We also suppose that the following bonds are sold on the market. Their face values are $100, they are emitted and reimbursed at par value, and coupon payments are annual.

Maturity	Coupon rate (annual)	Price
1 year and 4 months	4.0%	102.8
1 year and 5 months	4.50%	102.5
2 years	3.50%	98.3
3 years	4%	98.7
4 years	5%	101.6

1) Using the bootstrap method with linear interpolation, determine the spot rates for the maturities 4 months, 5 months, 1 year and 4 months, 1 year and 5 months, 2 years, 3 years and 4 years.

We obtain by interpolating

$$y(0,1/3) = 3.533\%,\ \ y(0,5/12) = 3.567\%.$$

Then, by the valuation formula for bonds

$$102.8 = \frac{4}{(1+y(0,1/3))^{1/3}} + \frac{104}{(1+y(0,4/3))^{4/3}}.$$

Hence $y(0,4/3) = 3.886\%$**.**

Similarly, we have

$$102.5 = \frac{4.5}{(1+y(0,5/12))^{5/12}} + \frac{104.5}{(1+y(0,17/12))^{17/12}}.$$

Hence $y(0,17/12) = 4.588\%$**.**

Similarly, we have

$$98.3 = \frac{3.5}{(1+y(0,1))^{1}} + \frac{103.5}{(1+y(0,2))^{2}}.$$

Hence $y(0,2) = 4.428\%$**.**

Similarly, we have

$$98.7 = \frac{4}{(1+y(0,1))^{1}} + \frac{4}{(1+y(0,2))^{2}} + \frac{104}{(1+y(0,3))^{3}}.$$

Hence $y(0,3) = 4.492\%$**.**

Similarly, we have

$$101.6 = \frac{5}{(1+y(0,1))^1} + \frac{5}{(1+y(0,2))^2} + \frac{5}{(1+y(0,3))^3} + \frac{105}{(1+y(0,4))^4}.$$

Hence $y(0,4) = 4.578\%$**.**

2) Represent the spot curve for maturities between 1 day and 4 years.