Introduction to Mathematical Finance
Dylan Possamaï

Assignment 10 (solutions)

A stochastic volatility model

Fix some positive integer T. We consider a T-period financial market defined as follows

$$\Omega := \big(\{1,-1\}\times\{1,-1\}\big)^T,\ \mathcal{F} := 2^\Omega.$$

We let (ε,η) be the canonical process on Ω, which simply means that for any $\omega := \big((\omega_t^1,\omega_t^2)\big)_{t\in\{1,\dots,T\}}$, we have

$$\varepsilon_t(\omega) := \omega_t^1,\ \eta_t(\omega) := \omega_t^2,\ t\in\{1,\dots,T\}.$$

The probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ is defined such that for any $t\in\{1,\dots,T\}$, the random variables η_t and ε_t are $\mathbb{Q}$-independent and centred, and such that the vector $(\eta_t,\varepsilon_t)_{t\in\{1,\dots,T\}}$ is i.i.d. under $\mathbb{Q}$.

The non-risky asset is constant, that is

$$S_t^0(\omega) := 1,\ (t,\omega)\in\{0,\dots,T\}\times\Omega,$$

and the unique risky asset satisfies that $S_0>0$ is given and

$$S_{t+1}(\omega) := \big(1+\sigma_{t+1}(\omega)\varepsilon_{t+1}(\omega)\big)S_t(\omega),\ \text{with } \sigma_{t+1}(\omega) := \bar\sigma+\theta\eta_{t+1}(\omega),\ (t,\omega)\in\{0,\dots,T-1\}\times\Omega,$$

where $(\bar\sigma,\theta)\in(0,+\infty)\times[0,+\infty)$ are such that $\bar\sigma+\theta<1$.

The filtration $\mathbb{F}$ on the market is then the one generated by S^1, that is

$$\mathcal{F}_0 := \{\Omega,\emptyset\},\ \mathcal{F}_t := \sigma\big((S_1,\dots,S_t)\big),\ t\in\{1,\dots,T\}.$$

1)a) Show that for any $t\in\{0,\dots,T\}$, we have $S_t>0$, $\mathbb{Q}$-a.s.

It is clear that for any $t\in\{0,\dots,T-1\}$

$$S_{t+1}\geq S_t\big(1-(\bar\sigma+\theta)\big),$$

which shows by an immediate induction that

$$S_t\geq S_0\big(1-(\bar\sigma+\theta)\big)^t>0,$$

since we assumed that $1>\bar\sigma+\theta$**.**

1)b) Show that for any $t\in\{1,\dots,T\}$, σ_t is $\mathcal{F}_t$-measurable, and deduce that whenever $\bar\sigma\neq\theta$, we have

$$\mathcal{F}_t = \sigma\big((\eta_1,\varepsilon_1,\dots,\eta_t,\varepsilon_t)\big),\ t\in\{1,\dots,T\},$$

and that in general

$$\mathcal{F}_t \subset \sigma\big((\eta_1,\varepsilon_1,\dots,\eta_t,\varepsilon_t)\big),\ t\in\{1,\dots,T\}.$$

First, if $\theta=0$, then for any $t\in\{1,\dots,T\}$, σ_t is always equal to $\bar\sigma$ and is thus immediately $\mathcal{F}_t$-measurable. When $\theta>0$, for any $t\in\{1,\dots,T\}$, recall that

$$\sigma_t = \bar\sigma+\theta\eta_t.$$

Hence

$$\{\sigma_t=\bar\sigma+\theta\} = \{\eta_t=1\} = \bigg\{\frac{S_t}{S_{t-1}}=1+\bar\sigma+\theta\bigg\}\bigcup\bigg\{\frac{S_t}{S_{t-1}}=1-\bar\sigma-\theta\bigg\},$$
$$\{\sigma_t=\bar\sigma-\theta\} = \{\eta_t=-1\} = \bigg\{\frac{S_t}{S_{t-1}}=1-\bar\sigma+\theta\bigg\}\bigcup\bigg\{\frac{S_t}{S_{t-1}}=1+\bar\sigma-\theta\bigg\}.$$

As long as $\bar{\sigma} \neq \theta$, all four values $1+\bar{\sigma}+\theta$, $1+\bar{\sigma}-\theta$, $1-\bar{\sigma}+\theta$, $1-\bar{\sigma}-\theta$ are different, meaning that η_t is indeed known as soon as the values of S_t and S_{t-1} are known, so that η_t is $\mathcal{F}_t$-measurable, and thus so is σ_t.

Even if $\bar{\sigma} = \theta$, we can still distinguish between the events $\{\eta_t = 1\}$ and $\{\eta_t = -1\}$ by simply knowing the values of S_t and S_{t-1}, since $1 \pm 2\bar{\sigma} \neq 1$.

Next, the inclusion is always clear by definition of S. In addition, when $\theta \neq \bar{\sigma}$, σ is never equal to 0, so that for any $t \in \{1, \dots, T\}$

$$\varepsilon_t = \frac{1}{\sigma_t}\bigg(\frac{S_t}{S_{t-1}} - 1\bigg),$$

and thus ε_t is also $\mathcal{F}_t$-measurable. This gives the desired remaining result.

1)c) Show that $\mathbb{Q}$ is a risk-neutral measure in this market and that (**NA**) holds.

Ω being finite, all random variables are bounded. We have then for any $t \in \{0, \dots, T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\big[S_{t+1}\big|\mathcal{F}_t\big] = S_t\Big(1 + \mathbb{E}^{\mathbb{Q}}\big[\sigma_{t+1}\varepsilon_{t+1}\big|\mathcal{F}_t\big]\Big).$$

We have seen in the previous question that

$$\mathcal{F}_t \subset \sigma\big((\eta_1, \varepsilon_1, \dots, \eta_t, \varepsilon_t)\big),\ t \in \{1, \dots, T\}.$$

This shows that for any $t \in \{0, \dots, T-1\}$, ε_{t+1} and σ_{t+1} are both independent of $\mathcal{F}_t$, and independent of each other. Hence, since these variables are also centred

$$\mathbb{E}^{\mathbb{Q}}\big[S_{t+1}\big|\mathcal{F}_t\big] = S_t\Big(1 + \mathbb{E}^{\mathbb{Q}}[\sigma_{t+1}]\mathbb{E}^{\mathbb{Q}}[\varepsilon_{t+1}]\Big) = S_t,$$

proving thus that the discounted value of S is an $(\mathbb{F}, \mathbb{Q})$-martingale, that $\mathbb{Q}$ is a risk-neutral measure, and that (NA) holds in this market.

1)d) Is $\mathbb{Q}$ the only risk-neutral measure in this market? Is the market complete?

From the computations in the previous question it is clear that a sufficient condition for a measure $\mathbb{Q}'$ on $(\Omega, \mathcal{F})$ to be risk-neutral is that for any $t \in \{1, \dots, T\}$, the random variables η_t and ε_t are $\mathbb{Q}'$-independent, that *at least one of them* is centred, and such that the vector $(\eta_t, \varepsilon_t)_{t \in \{1, \dots, T\}}$ is i.i.d. under $\mathbb{Q}'$. This gives immediately infinitely many additional risk-neutral measures, and the market cannot be complete.

2) We are now given a European option with maturity T and payoff $h(S_T)$ for some map $h : (0, +\infty) \longrightarrow \mathbb{R}$. Despite the market being incomplete, we decide to use as a viable price for this option its no-arbitrage price under the risk-neutral measure $\mathbb{Q}$, which we denote by $(V_t)_{t \in \{0, \dots, T\}}$.

Show that there exists a map $v : \{0, \dots, T\} \times (0, +\infty)$ such that

$$V_t = v(t, S_t),\ \mathbb{Q}\text{-a.s.},$$

and that v is defined through the following backward induction, for $(t, x) \in \{1, \dots, T\} \times (0, +\infty)$

$$\begin{aligned} v(T, x) &:= h(x), \\ v(t-1, x) &:= \frac{1}{4}\Big(v\big(t, x(1+\bar{\sigma}+\theta)\big) + v\big(t, x(1+\bar{\sigma}-\theta)\big) + v\big(t, x(1-\bar{\sigma}+\theta)\big) + v\big(t, x(1-\bar{\sigma}-\theta)\big)\Big). \end{aligned}$$

We will show the result by induction as usual. For $t = T$, the result is obvious. Let us assume it is true for some $t \in \{1, \dots, T\}$, then

$$V_{t-1} = \mathbb{E}^{\mathbb{Q}}[h(S_T)|\mathcal{F}_{t-1}] = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}[h(S_T)|\mathcal{F}_t]\Big|\mathcal{F}_{t-1}\Big] = \mathbb{E}^{\mathbb{Q}}[V_t|\mathcal{F}_{t-1}] = \mathbb{E}^{\mathbb{Q}}[v(t, S_t)|\mathcal{F}_{t-1}].$$

The formula is then immediate by noting that given $\mathcal{F}_{t-1}$, the product $\varepsilon_t\sigma_t$ takes four values, each with probability $1/4$, since ε_t and η_t are $\mathbb{Q}$-independent of each other and of $\mathcal{F}_{t-1}$, are centred, and take values in $\{1, -1\}$.

3)a) Show that for any self-financing portfolio $\Delta \in \mathcal{A}(\mathbb{R})$, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta}\big)^2\Big] = \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1} - V_t - \Delta_t(S_{t+1} - S_t)\big)^2\big].$$

Notice that both V and $X^{V_0,\Delta}$ are $(\mathbb{F},\mathbb{Q})$-martingales (again integrability is immediate since Ω is finite). Hence, $M := V - X^{V_0,\Delta} + V_0$ is also an $(\mathbb{F},\mathbb{Q})$-martingale, which is square-integrable since Ω is finite. We thus have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta}\big)^2\Big] = \mathbb{E}^{\mathbb{Q}}\big[(M_T - M_0)^2\big] = \mathbb{E}^{\mathbb{Q}}\big[M_T^2 - M_0^2\big] &= \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\big[M_{t+1}^2 - M_t^2\big] \\ &= \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\big[(M_{t+1} - M_t)^2\big] \\ &= \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1} - V_t - \Delta_t(S_{t+1} - S_t)\big)^2\big].\end{aligned}$$

3)b) Show that for any $t \in \{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1}-V_t-\Delta_t(S_{t+1}-S_t)\big)^2\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)^2\big|\mathcal{F}_t\big]-2\Delta_t\mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)(S_{t+1}-S_t)\big|\mathcal{F}_t\big]+\Delta_t^2\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1}-S_t)^2\big|\mathcal{F}_t\big].$$

It is obvious by expanding the square and using the fact that Δ_t is $\mathcal{F}_t$-measurable.

3)c) For any $t \in \{0,\dots,T-1\}$, compute $\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1} - S_t)^2\big|\mathcal{F}_t\big]$ and show that this is always a positive quantity.

We have, since ε^2 and η^2 are constant equal to 1

$$\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1} - S_t)^2\big|\mathcal{F}_t\big] = S_t^2\mathbb{E}^{\mathbb{Q}}\big[\sigma_{t+1}^2\varepsilon_{t+1}^2\big|\mathcal{F}_t\big] = S_t^2\Big(\bar\sigma^2 + \theta^2 + 2\bar\theta\sigma\mathbb{E}^{\mathbb{Q}}[\eta_{t+1}|\mathcal{F}_t]\Big) = S_t^2(\bar\sigma^2 + \theta^2) > 0,$$

where we used that η_t is centred under $\mathbb{Q}$, and that η and ε are $\mathbb{Q}$-independent.

3)d) Define

$$\Delta_t^\star := \frac{\mathbb{E}^{\mathbb{Q}}\big[(V_{t+1} - V_t)(S_{t+1} - S_t)\big|\mathcal{F}_t\big]}{\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1} - S_t)^2\big|\mathcal{F}_t\big]},\ t \in \{0,\dots,T-1\}.$$

Show that for any $t \in \{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \Delta_t^\star(S_{t+1} - S_t)\big)^2\Big|\mathcal{F}_t\Big] = \min\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \zeta(S_{t+1} - S_t)\big)^2\Big|\mathcal{F}_t\Big] : \zeta\ \mathbb{R}\text{-valued and } \mathcal{F}_t\text{-measurable}\Big\},$$

and then that for any $t \in \{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \Delta_t^\star(S_{t+1} - S_t)\big)^2\Big] = \min\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \zeta(S_{t+1} - S_t)\big)^2\Big] : \zeta\ \mathbb{R}\text{-valued and } \mathcal{F}_t\text{-measurable}\Big\},$$

For any ζ which is $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1}-V_t-\zeta(S_{t+1}-S_t)\big)^2\Big|\mathcal{F}_t\Big] = \mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)^2\big|\mathcal{F}_t\big]-2\zeta\mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)(S_{t+1}-S_t)\big|\mathcal{F}_t\big]+\zeta^2\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1}-S_t)^2\big|\mathcal{F}_t\big].$$

Now, for any $\omega \in \Omega$, the right-hand side above is a second-order polynomial in $\zeta^2(\omega)$, which clearly attains its infimum at $\Delta_t^\star(\omega)$ given in the question, thanks to the positivity shown in 3)c). This shows the first result. Next, notice that for any ζ which is $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \zeta(S_{t+1} - S_t)\big)^2\Big] &= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1} - V_t - \zeta(S_{t+1} - S_t)\big)^2\big|\mathcal{F}_t\big]\Big] \\ &\geq \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1} - V_t - \Delta_t^\star(S_{t+1} - S_t)\big)^2\big|\mathcal{F}_t\big]\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \Delta_t^\star(S_{t+1} - S_t)\big)^2\Big].\end{aligned}$$

This proves that

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1}-V_t-\Delta_t^\star(S_{t+1}-S_t)\big)^2\Big] \leq \min\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1}-V_t-\zeta(S_{t+1}-S_t)\big)^2\Big] : \zeta\ \mathbb{R}\textbf{-valued and } \mathcal{F}_t\textbf{-measurable}\Big\},$$

and we conclude by noticing that choosing $\Delta_t^\star$, the infimum is then automatically attained.

3)e) Conclude that

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = \min\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta}\big)^2\Big] : \Delta \in \mathcal{A}(\mathbb{R})\Big\}.$$

We have by 3)a) and 3)d) that for any $\Delta \in \mathcal{A}(\mathbb{R})$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta}\big)^2\Big] = \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1}-V_t-\Delta_t(S_{t+1}-S_t)\big)^2\big] &\geq \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{Q}}\big[\big(V_{t+1}-V_t-\Delta_t^\star(S_{t+1}-S_t)\big)^2\big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big],\end{aligned}$$

from which we deduce

$$\min\Big\{\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta}\big)^2\Big] : \Delta \in \mathcal{A}(\mathbb{R})\Big\} \geq \mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big],$$

and we recover equality since $\Delta^\star \in \mathcal{A}(\mathbb{R})$.

3)f) Show that for any $t \in \{0,\ldots,T-1\}$

$$\Delta_t^\star = \varphi(t,S_t),\ \text{where } \varphi(t,x) := \frac{\mathbb{E}^{\mathbb{Q}}\big[\sigma_1\varepsilon_1\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)\big]}{x(\bar\sigma^2+\theta^2)},\ x \in (0,+\infty).$$

We start from the formula defining $\Delta^\star$, and have for any $t \in \{0,\ldots,T-1\}$, using 3)c)

$$\begin{aligned}\Delta_t^\star = \frac{\mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)(S_{t+1}-S_t)\big|\mathcal{F}_t\big]}{\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1}-S_t)^2\big|\mathcal{F}_t\big]} &= \frac{\mathbb{E}^{\mathbb{Q}}\big[(v(t+1,S_{t+1})-v(t,S_t))(S_{t+1}-S_t)\big|\mathcal{F}_t\big]}{S_t^2(\bar\sigma^2+\theta^2)}\\ &= \frac{\mathbb{E}^{\mathbb{Q}}\big[(v(t+1,S_t(1+\sigma_{t+1}\varepsilon_{t+1}))-v(t,S_t))\sigma_{t+1}\varepsilon_{t+1}\big|\mathcal{F}_t\big]}{S_t(\bar\sigma^2+\theta^2)},\end{aligned}$$

and we conclude using the fact that under $\mathbb{Q}$, the random variables $(\sigma_t,\varepsilon_t)_{t\in\{1,\ldots,T\}}$ are i.i.d. and independent of $\mathcal{F}_t$.

4)a) We define for any $(t,x) \in \{0,\ldots,T-1\}\times(0,+\infty)$

$$\mathcal{R}(t,x) := \mathbb{E}^{\mathbb{Q}}\Big[\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)^2\Big] - \frac{\big(\mathbb{E}^{\mathbb{Q}}\big[\sigma_1\varepsilon_1\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)\big]\big)^2}{\bar\sigma^2+\theta^2}.$$

Show that $\mathcal{R} \geq 0$.

This is an immediate application of Cauchy–Schwarz's inequality. Indeed

$$\begin{aligned}\Big(\mathbb{E}^{\mathbb{Q}}\big[\sigma_1\varepsilon_1\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)\big]\Big)^2 &\leq \mathbb{E}^{\mathbb{Q}}\Big[\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)^2\Big]\mathbb{E}^{\mathbb{Q}}\big[\sigma_1^2\varepsilon_1^2\big]\\ &= (\bar\sigma^2+\theta^2)\mathbb{E}^{\mathbb{Q}}\Big[\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)^2\Big].\end{aligned}$$

4)b) Show that

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\big[\mathcal{R}(t,S_t)\big].$$

With computations similar to the ones from 3)f), we have that

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(V_{t+1} - V_t - \Delta_t^\star(S_{t+1}-S_t)\big)^2\Big|\mathcal{F}_t\Big] = \mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)^2\big|\mathcal{F}_t\big] - \frac{\big(\mathbb{E}^{\mathbb{Q}}\big[(V_{t+1}-V_t)(S_{t+1}-S_t)\big|\mathcal{F}_t\big]\big)^2}{\mathbb{E}^{\mathbb{Q}}\big[(S_{t+1}-S_t)^2\big|\mathcal{F}_t\big]} = \mathcal{R}(t,S_t).$$

The result is then immediate from 3)a) and 3)d).

4)c) Show that $\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = 0$ for any choice of the payoff h if and only if $\theta = 0$.

Hint: for the direct part of the equivalence, it may prove useful to first establish the result when $T=1$, and to think about choosing a specific payoff h. Then to see how this reasoning can be modified to work for T arbitrary.

Assume first that $\theta = 0$. Then the process σ is constant equal to $\bar\sigma$, and since the process ε is centred under $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)^2\Big] = \frac{1}{2}\bigg(\frac{v(t,x(1+\bar\sigma)) - v(t,x(1-\bar\sigma))}{2}\bigg)^2 + \frac{1}{2}\bigg(\frac{v(t,x(1-\bar\sigma)) - v(t,x(1+\bar\sigma))}{2}\bigg)^2$$
$$= \bigg(\frac{v(t,x(1+\bar\sigma)) - v(t,x(1-\bar\sigma))}{2}\bigg)^2.$$

Similarly

$$\mathbb{E}^{\mathbb{Q}}\big[\sigma_1\varepsilon_1\big(v(t+1,x(1+\sigma_1\varepsilon_1)) - v(t,x)\big)\big] = \frac{\bar\sigma}{2}\big(v(t,x(1+\bar\sigma)) - v(t,x(1-\bar\sigma))\big),$$

so that in this case $\mathcal{R}(t,x) = 0$, and we conclude from 4)b that $\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = 0$, whatever we choose for h.

Conversely, if $\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = 0$, then again by 4)$b$), we must have, since $\mathcal{R}$ is non-negative, that $\mathbb{E}^{\mathbb{Q}}\big[\mathcal{R}(t,S_t)\big] = 0$ for any $t \in \{0,\dots,T-1\}$. Recalling how we showed the non-negativity of $\mathcal{R}$, this can only happen in the equality case of Cauchy–Schwarz's inequality, and therefore there must exist some $\lambda(t,x) \in \mathbb{R}$ such that

$$v\big(t+1,x(1+\sigma_1\varepsilon_1)\big) - v(t,x) = \gamma(t,x)\sigma_1\varepsilon_1.$$

In particular, if we follow the hint and consider the one-period case $T=1$, and choose $x = S_0$

$$h\big(S_0(1+\sigma_1\varepsilon_1)\big) = v\big(1,S_0(1+\sigma_1\varepsilon_1)\big) = v(0,S_0) + \gamma(0,S_0)\sigma_1\varepsilon_1. \tag{0.1}$$

We claim that this equality is impossible when $\theta > 0$, for an appropriate choice of the payoff h. Indeed, the product $\varepsilon_1\sigma_1$ takes at least three different values (usually 4, unless $\bar\sigma = \theta > 0$). Hence, the right-hand side of the previous inequality takes either a unique value when $\gamma(0,S_0) = 0$, or at least three distinct ones when $\gamma(0,S_0) \neq 0$. Now for the left-hand side, take

$$h(y) := \big(y - S_0(1+\bar\sigma\vee\theta)\big)^+,\ y \geq 0.$$

Then it is immediate that $h\big(S_0(1+\sigma_1\varepsilon_1)\big)$ takes exactly two values, namely 0 and $S_0(\bar\sigma\wedge\theta)$. Hence for this choice of h, Equation (0.1) cannot hold, and we thus must have $\theta = 0$.

Now in the general case with $T > 1$, we can actually do the exact same reasoning at the last period $T-1$ and T, and taking this time $x = S_0(1+\bar\sigma+\theta)^{T-1}$, and the payoff $h(y) := \big(y - S_0(1+\bar\sigma+\theta)^{T-1}(1+\bar\sigma\vee\theta)\big)^+$.

4)d) How do you interpret the result of the previous question? What does the quantity $\mathbb{E}^{\mathbb{Q}}\big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\big]$ represent?

The result of the previous section stipulates that we can replicate any contingent claim in this market, which is then complete (indeed, having $\mathbb{E}^{\mathbb{Q}}\Big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\Big] = 0$ is equivalent to saying that we can replicate the option with payoff h) if and only if $\theta = 0$, which corresponds to being in a binomial model. In that sense $\mathbb{E}^{\mathbb{Q}}\big[\big(V_T - X_T^{V_0,\Delta^\star}\big)^2\big]$ is a sort of quadratic hedging error.

Doubling strategies

We consider here a binomial model with one risky asset, but where the time horizon T is now equal to $+\infty$, so that Ω is now the set of sequences $(\omega_n)_{n\in\mathbb{N}\setminus\{0\}}$, where $\omega_n \in \{\omega^u, \omega^d\}$ for any positive integer n.

For simplicity, we fix the value of the non-risky asset to 1, that its $S_t^0 = 1$, for any $t \in \mathbb{N}$, and we still have that $S_0 > 0$ is given and

$$S_{t+1}(\omega) = uS_t(\omega)\mathbf{1}_{\{\omega_{t+1}=\omega^u\}} + dS_t(\omega)\mathbf{1}_{\{\omega_{t+1}=\omega^d\}},\ (t,\omega) \in \mathbb{N}\times\Omega,$$

where $0 < d < 1 < u$. We take the filtration $\mathbb{F}$ defined by

$$\mathcal{F}_0 := \{\Omega, \emptyset\},\ \mathcal{F}_t := \sigma\big((S_1, \dots, S_t)\big),\ t \in \mathbb{N}\setminus\{0\},$$

and impose $\mathcal{F} := \sigma\big(\cup_{t\in\mathbb{N}}\mathcal{F}_t\big)$. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is defined without further comments for now

The notion of self-financing strategies is directly extended to this setting by imposing that the consumption process is 0 at any time $t \in \mathbb{N}\setminus\{0\}$, and we use the same notations for the associated wealth processes as in the Lecture Notes. As for the question of arbitrage opportunities, we need the following generalised notion.

Definition 0.1. *A generalised arbitrage opportunity $\Delta \in \mathcal{A}(\mathbb{R})$ is such that there exists an $\mathbb{F}$-stopping time τ, with $\mathbb{P}[\tau < +\infty] = 1$ as well as*

$$\mathbb{P}\big[X_\tau^{0,\Delta} \geq 0\big] = 1, \text{ and } \mathbb{P}\big[X_\tau^{0,\Delta} > 0\big] > 0.$$

We will consider throughout the problem the so-called 'doubling' strategy $\mathbf{\Delta}$, which satisfies

$$\mathbf{\Delta}_0 := 1,\ \mathbf{\Delta}_t := \left(1 + \frac{u}{d}\right)\mathbf{\Delta}_{t-1}\mathbf{1}_{\{\omega_t=\omega^d\}}.$$

1)a) Check that $\mathbf{\Delta} \in \mathcal{A}(\mathbb{R})$ and give an interpretation for that strategy (it could prove useful to see what happens with specific values of u, d and S_0).

The fact that $\mathbf{\Delta}$ is $\mathbb{F}$-adapted is obvious by definition, since $\mathbf{\Delta}_0$ is a constant and $\mathbf{\Delta}$ is defined recursively. Let us fix for instance $u = 1.1$ and $d = 0.9$, and that $S_0 = 100$. Then the strategy proceeds as follows

- **at $t = 0$, we borrow 100 in cash to acquire one unit of the risky asset;**

- **at $t = 1$, if $\omega_1 = \omega^u$, we sell our risky asset, obtain 110, pay back the 100 we borrowed, and stop trading, having gained 10;**

- **at $t = 1$, if $\omega_1 = \omega^d$, we borrow $u/d \times dS_0 = 110$ to buy $u/d = 11/9$ additional risky assets;**

- **at $t = 2$, if $\omega_1 = \omega^d$, and $\omega_2 = \omega^u$, we sell our $1 + 11/9$ risky assets to obtain 220 and pay back $100 + 110 = 210$ we borrowed at time 0 and 1,**

and if $\omega_1 = \omega_2 = \omega^d$, this keeps going on as long as there isn't an upward move for the risky asset price. In a nutshell, the strategy consists in borrowing more and more cash to acquire risky assets until the first time a favourable event occurs in the market, and the risky asset price increases.

$1)b)$ Let $(\eta_t)_{t\in\mathbb{N}}$ be the process representing the number of non-risky assets held in the self-financing portfolio $(0,\mathbf{\Delta})$. Show that for any $\omega\in\Omega$

$$\eta_0(\omega) = -S_0,\ \eta_t(\omega) = \begin{cases} \eta_{t-1}(\omega) - u\mathbf{\Delta}_{t-1}(\omega)S_{t-1}(\omega), \text{ if } \omega_t = \omega^d, \\ \eta_{t-1}(\omega) + u\mathbf{\Delta}_{t-1}(\omega)S_{t-1}(\omega), \text{ if } \omega_t = \omega^u. \end{cases}$$

The value for η_0 is immediate since $0 = \eta_0 + 1\times S_0$. Moreover, the general formula from the lectures gives us that for any $t\in\mathbb{N}\setminus\{0\}$

$$\eta_t = \eta_{t-1} - (\mathbf{\Delta}_t - \mathbf{\Delta}_{t-1})S_t = \eta_{t-1} - \left(\left(1+\frac{u}{d}\right)\mathbf{1}_{\{\omega_t=\omega^d\}} - 1\right)\mathbf{\Delta}_{t-1}S_t,$$

from which the result is immediate.

$1)c)$ Deduce that for any $t\in\mathbb{N}\setminus\{0\}$, we have on the event $\{\omega_1 = \cdots = \omega_t = \omega^d\}$

$$\mathbf{\Delta}_t = \left(1+\frac{u}{d}\right)^t,\ \eta_t = -S_0\left(1+\frac{u}{u+d-1}\big((u+d)^t - 1\big)\right).$$

On that event, $(\mathbf{\Delta}_s)_{s\in\{0,\dots,t\}}$ is a geometric sequence, so that the first formula is obvious. Using the previous question we also have for any $s\in\{1,\dots,t\}$

$$\eta_s = \eta_{s-1} - u\mathbf{\Delta}_{s-1}S_{s-1} = \eta_{s-1} - u\left(1+\frac{u}{d}\right)^{s-1}d^{s-1}S_0,$$

from which we deduce

$$\eta_t = \eta_0 - uS_0\sum_{s=1}^t (u+d)^{s-1} = -S_0\left(1+u\frac{(u+d)^t-1}{u+d-1}\right).$$

$2)a)$ Show that for any $t\in\mathbb{N}$, we have on the event $\{\omega_1 = \cdots = \omega_t = \omega^d\}$

$$X_t^{0,\mathbf{\Delta}} = -\frac{S_0|1-d|}{u+d-1}\big((u+d)^t - 1\big).$$

We use the general formula

$$X_t^{0,\mathbf{\Delta}} = \sum_{s=0}^{t-1}\mathbf{\Delta}_s\big(S_{s+1}-S_s\big) = S_0(d-1)\sum_{s=0}^{t-1}(u+d)^s = S_0(d-1)\frac{(u+d)^t-1}{u+d-1},$$

which is the desired result.

$2)b)$ Show that for any $t\in\mathbb{N}\setminus\{0\}$, we have on the event $\{\omega_1 = \cdots = \omega_{t-1} = \omega^d\}\cap\{\omega_t = \omega^u\}$

$$X_t^{0,\mathbf{\Delta}} = X_{t-1}^{0,\mathbf{\Delta}} + S_0(u-1)(u+d)^{t-1},$$

and then that on the same event

$$X_t^{0,\mathbf{\Delta}} = \frac{S_0}{u+d-1}\big(|1-d| + u(u+d-2)(u+d)^{t-1}\big).$$

We have on this event

$$X_t^{0,\mathbf{\Delta}} = X_{t-1}^{0,\mathbf{\Delta}} + \mathbf{\Delta}_{t-1}(S_t - S_{t-1}) = X_{t-1}^{0,\mathbf{\Delta}} + \left(1+\frac{u}{d}\right)^{t-1}(u-1)d^{t-1}S_0,$$

so that using the previous question

$$X_t^{0,\mathbf{\Delta}} = -\frac{S_0|1-d|}{u+d-1}\big((u+d)^{t-1} - 1\big) + S_0(u-1)(u+d)^{t-1},$$

and the result follows.

3) From now on, we assume that

$$u + d \geq 2,$$

and that $\mathbb{P}$ is such that $\mathbb{P}\big[S_{t+1}/S_t = d,\ \forall t \in \mathbb{N}\big] = 0$.

We also define

$$\tau(\omega) := \min\big\{t \in \mathbb{N} \setminus \{0\} : \omega_t = \omega^u\big\}.$$

3)a) Show that τ is an $\mathbb{F}$-stopping time and that $\mathbb{P}[\tau < +\infty] = 1$.

It is immediate that for any $t \in \mathbb{N} \setminus \{0\}$

$$\{\tau = t\} = \{\omega_1 = \cdots = \omega_{t-1} = \omega^d\} \cap \{\omega_t = \omega^u\} = \left\{\frac{S_1}{S_0} = \cdots = \frac{S_{t-1}}{S_{t-2}} = d\right\} \bigcap \left\{\frac{S_t}{S_{t-1}} = u\right\},$$

which is obviously an element of $\mathcal{F}_t$. Besides

$$\mathbb{P}[\tau = +\infty] = \mathbb{P}\big[S_{t+1}/S_t = d,\ \forall t \in \mathbb{N}\big] = 0,$$

which proves the second claim.

3)b) Show that for any self-financing strategy $(x, \Delta) \in \mathbb{R} \times \mathcal{A}(\mathbb{R})$ the 'stopped' strategy (x, Δ^τ), with $\Delta_t^\tau(\omega) := \Delta_{t \wedge \tau(\omega)}(\omega)$ is still admissible and self-financing.

This follows from Lemma B.2.8 in the Lecture Notes, which says that the stopped process Δ^τ is $\mathbb{F}$-adapted whenever Δ is. Hence $\Delta^\tau \in \mathcal{A}(\mathbb{R})$. The fact that it remains self-financing is also immediate by definition, since using the fact that $\boldsymbol{\Delta}$ is self-financing, and that $\Delta_\tau = 0$

$$\begin{aligned}\boldsymbol{\Delta}_{s+1}^\tau S_{s+1} = \boldsymbol{\Delta}_{s+1} S_{s+1} \mathbf{1}_{\{\tau \geq s+1\}} + \boldsymbol{\Delta}_\tau S_{s+1} \mathbf{1}_{\{\tau \leq s\}} &= \boldsymbol{\Delta}_s S_{s+1} \mathbf{1}_{\{\tau \geq s+1\}} + \boldsymbol{\Delta}_\tau S_{s+1} \mathbf{1}_{\{\tau \leq s\}} \\ &= \boldsymbol{\Delta}_s^\tau S_{s+1} - \boldsymbol{\Delta}_s S_{s+1} \mathbf{1}_{\{\tau = s\}} = \boldsymbol{\Delta}_s^\tau S_{s+1}.\end{aligned}$$

3)c) Show that $\boldsymbol{\Delta}^\tau$ verifies

$$\boldsymbol{\Delta}_t^\tau = \mathbf{1}_{\{t+1 \leq \tau\}} \boldsymbol{\Delta}_t,\ t \in \mathbb{N}.$$

Deduce that

$$\big(X_t^{0,\boldsymbol{\Delta}}\big)^\tau = X_t^{0,\boldsymbol{\Delta}^\tau},\ t \in \mathbb{N}.$$

The equality is obvious by definition on the event $\{\tau \leq t\}$ as both sides are equal to 0. On the event $\{\tau \geq t+1\}$, this is again obvious. We thus have

$$\big(X_t^{0,\boldsymbol{\Delta}}\big)^\tau = X_{t\wedge\tau}^{0,\boldsymbol{\Delta}} = \boldsymbol{\Delta}_{t\wedge\tau} S_{t\wedge\tau} = \boldsymbol{\Delta}_t^\tau S_{t\wedge\tau} = \boldsymbol{\Delta}_t^\tau S_t \mathbf{1}_{\{t+1 \leq \tau\}} + \boldsymbol{\Delta}_t^\tau S_\tau \mathbf{1}_{\{\tau \leq t\}} = \boldsymbol{\Delta}_t^\tau S_t \mathbf{1}_{\{t+1 \leq \tau\}} = \boldsymbol{\Delta}_t^\tau S_t = X_t^{0,\boldsymbol{\Delta}^\tau}.$$

3)d) Prove that $\boldsymbol{\Delta}^\tau$ is a generalised arbitrage opportunity.

We have $\boldsymbol{\Delta}^\tau \in \mathcal{A}(\mathbb{R})$ and it is self-financing by the previous questions. Moreover, we have $\mathbb{P}[\tau < +\infty] = 1$ and

$$X_\tau^{0,\boldsymbol{\Delta}^\tau} = X_\tau^{0,\boldsymbol{\Delta}} = \frac{S_0}{u+d-1}\big(|1-d| + u(u+d-2)(u+d)^{\tau-1}\big),$$

which is $\mathbb{P}$–a.s. positive since $u + d \geq 2$ and $d < 1$.

4) We now assume that $\mathbb{P}$ is constructed so that the sequence $(S_{t+1}/S_t)_{t\in\mathbb{N}}$ is constituted of $\mathbb{P}$-independent and identically distributed random variables with mean 1 (under $\mathbb{P}$ of course).

4)a) Show that $\mathbb{P}$ is a risk-neutral measure.

The independence assumption ensures that (recall that the filtration is generated by S)

$$\mathbb{E}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t] = S_t\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\middle|\mathcal{F}_t\right] = S_t\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\right] = S_t,$$

thus proving the martingale property (integrability is obvious here). Hence the result.

4)b) Show that under $\mathbb{P}$, the distribution of τ is a geometric distribution with a parameter λ[1] you will give explicitly in terms of u and b.

Deduce that in this context, $\mathbf{\Delta}^\tau$ is indeed a generalised arbitrage opportunity. What can you deduce concerning the possibility of extending the first FTAP to a setting with infinite horizon?

We use again independence

$$\mathbb{P}[\tau = t] = \mathbb{P}\Big[\big\{\omega_1 = \cdots = \omega_{t-1} = \omega^d\big\} \cap \{\omega_t = \omega^u\}\Big] = \big(1 - \mathbb{P}[\omega_1 = \omega^u]\big)^{t-1}\mathbb{P}[\omega_1 = \omega^u].$$

Moreover

$$1 = \mathbb{E}^{\mathbb{P}}\left[\frac{S_1}{S_0}\right] = u\mathbb{P}[\omega_1 = \omega^u] + d\big(1 - \mathbb{P}[\omega_1 = \omega^u]\big),$$

so that $\mathbb{P}[\omega_1 = \omega^u] = \frac{1-d}{u-d}$, as usual for binomial models, and the parameter of the geometric distribution is $\lambda = \frac{1-d}{u-d}$. This proves that in this setting $\mathbb{P}[\tau < +\infty] = 1$ and that we can apply the results of 3) to deduce the existence of a generalised arbitrage opportunity.

Overall, we constructed an infinite horizon financial market in which a risk-neutral measure existed, but there was nonetheless a generalised arbitrage opportunity. This proves that an extension of the first FTAP to infinite horizon is not straightforward.

4)c) Show that when $u \geq 2$

$$\mathbb{E}^{\mathbb{P}}\big[X^{0,\Delta^\tau}_{\tau-1}\big] = -\infty.$$

What can you deduce concerning the practical implementability of this strategy?

We have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X^{0,\Delta^\tau}_{\tau-1}\big] = \mathbb{E}^{\mathbb{P}}\big[X^{0,\Delta}_{\tau-1}\big] = -\frac{S_0|1-d|}{u+d-1}\big(\mathbb{E}^{\mathbb{P}}\big[(u+d)^{\tau-1}\big] - 1\big) &= -\frac{S_0|1-d|}{u+d-1}\left(\lambda\sum_{k=1}^{+\infty}(u+d)^{k-1}(1-\lambda)^{k-1} - 1\right)\\ &= -\frac{S_0|1-d|}{u+d-1}\left(\lambda\sum_{k=1}^{+\infty}(u-1)^{k-1} - 1\right) = -\infty,\end{aligned}$$

for $u \geq 2$. This means that in order to implement this strategy, we would on average have a wealth of $-\infty$ right before we finally make a gain and quit trading: this is therefore a strategy that requires the possibility of borrowing unbounded quantities of cash, which is of course not realistic in practice.

[1] That is

$$\mathbb{P}[\tau = t] = \lambda(1-\lambda)^{t-1},\ t \in \mathbb{N}\setminus\{0\}.$$