Introduction to Mathematical Finance
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Assignment 11 (solutions)

Relative entropy

For probability measures $\mathbb{Q}$ and $\mathbb{P}$ such that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, the (*relative*) *entropy* of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as

$$H(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{P}}\bigg[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\log\bigg(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg)\bigg].$$

In this problem, we consider the one-period trinomial market with $R=1$, $u_1+u_3=2$, and $u_2=1$ (see Section 2.3.2.3).

1) Find the measure $\mathbb{Q}^\star$ minimising the relative entropy $H(\mathbb{Q}|\mathbb{P})$ over all equivalent martingale measures $\mathbb{Q}\in\mathcal{M}(S)$.

We recall from the Lecture notes that the set $\mathcal{M}(S)$ is given here by

$$\mathbb{Q}[\{\omega^1\}] = q_1,\ \mathbb{Q}[\{\omega^2\}] = \frac{u_3-R}{u_3-u_2} - \frac{u_3-u_1}{u_3-u_2}q_1,\ \mathbb{Q}[\{\omega^3\}] = \frac{u_2-u_1}{u_3-u_2}q_1 + \frac{R-u_2}{u_3-u_2},$$

with $q_1\in\Big(\frac{(u_2-R)^+}{u_2-u_1},\frac{u_3-R}{u_3-u_1}\Big)$.

With the specific assumptions made here, this simplifies to

$$\mathbb{Q}[\{\omega^1\}] = q_1,\ \mathbb{Q}[\{\omega^2\}] = 1-2q_1,\ \mathbb{Q}[\{\omega^3\}] = q_1,$$

with $q_1\in\big(0,\frac{1}{2}\big)$.

The density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ **is then given by**

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega_1) := \frac{q_1}{p_1},\ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega_2) := \frac{1-2q_1}{p_2},\ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega_3) := \frac{q_1}{1-p_1-p_2}.$$

Hence, the entropy is given by

$$H(\mathbb{Q}|\mathbb{P}) = 2q_1\log\bigg(\frac{q_1}{\sqrt{p_1(1-p_1-p_2)}}\bigg) + (1-2q_1)\log\bigg(\frac{1-2q_1}{p_2}\bigg) =: g(q_1).$$

We differentiate and find

$$g'(q_1) = 2\log\bigg(\frac{q_1}{1-2q_1}\frac{p_2}{\sqrt{p_1(1-p_1-p_2)}}\bigg)$$

Let $\lambda := \frac{p_2}{\sqrt{p_1(1-p_1-p_2)}}$**,** g **attains its maximum on** $(0,1/2)$ **at** $1/(2+\lambda)$**, so that the risk-neutral** $\mathbb{Q}^\star$ **maximising the relative entropy is such that**

$$\mathbb{Q}^\star[\{\omega^1\}] = \frac{1}{2+\lambda},\ \mathbb{Q}[\{\omega^2\}] = \frac{\lambda}{2+\lambda},\ \mathbb{Q}[\{\omega^3\}] = \frac{1}{2+\lambda}.$$

2) Find the strategy $\Delta^\star\in\mathcal{A}(\mathbb{R})$ maximising the expected utility of final wealth, with initial wealth 0 and exponential utility with parameter $\alpha>0$, i.e.,

$$U(x) := \frac{1-\mathrm{e}^{-\alpha x}}{\alpha},\ \text{and } u(x)=0.$$

Verify that

$$\frac{\mathrm{d}\mathbb{Q}^\star}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)}}{\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)}\big]}.$$

We are trying to solve the problem

$$\sup_{\Delta\in\mathbb{R}} \mathbb{E}^{\mathbb{P}}\big[U(X_1^{0,\Delta})\big] = \sup_{\Delta\in\mathbb{R}} \Big\{ \underbrace{p_1 U\big(\Delta S_0(u_1-1)\big) + p_2 U(0) + (1-p_1-p_2)U\big(\Delta S_0(1-u_1)\big)}_{=:f(\Delta)} \Big\}.$$

We have

$$\begin{aligned} f'(\Delta) &= S_0(u_1-1)\Big(p_1 U'\big(\Delta S_0(u_1-1)\big) - (1-p_1-p_2)U'\big(\Delta S_0(1-u_1)\big)\Big) \\ &= S_0(u_1-1)\mathrm{e}^{-\alpha\Delta S_0(u_1-1)}\Big(p_1 - (1-p_1-p_2)\mathrm{e}^{2\alpha\Delta S_0(u_1-1)}\Big) \end{aligned}$$

Recalling that $u_1 < u_2 = 1$, we deduce that the supremum of f is attained at

$$\Delta^\star := -\frac{1}{2\alpha(1-u_1)S_0}\log\bigg(\frac{p_1}{1-p_1-p_2}\bigg).$$

Hence

$$\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)} = \bigg(\frac{p_1}{1-p_1-p_2}\bigg)^{\frac{S_1-S_0}{2(1-u_1)S_0}},\ \textbf{and}\ \mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)}\big] = p_2 + 2\sqrt{p_1(1-p_1-p_2)}.$$

Let $Y := \frac{\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)}}{\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-\alpha\Delta^\star(S_1-S_0)}]}$, we then have

$$Y(\omega_1) = \frac{1}{p_1(2+\lambda)} = \frac{\mathrm{d}\mathbb{Q}^\star}{\mathrm{d}\mathbb{P}}(\omega_1),\ Y(\omega_2) = \frac{\lambda}{p_2(2+\lambda)} = \frac{\mathrm{d}\mathbb{Q}^\star}{\mathrm{d}\mathbb{P}}(\omega_2),\ Y(\omega_3) = \frac{1}{(1-p_1-p_2)(2+\lambda)} = \frac{\mathrm{d}\mathbb{Q}^\star}{\mathrm{d}\mathbb{P}}(\omega_3),$$

proving the desired equality.

Mean–variance hedging

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\ldots,T\}}$. There is only one risky asset, and the non-risky asset value is constant equal to 1. Suppose also that

$$\mathbb{E}^{\mathbb{P}}[(S_t - S_{t-1})^2|\mathcal{F}_{t-1}] < +\infty,\ \mathbb{P}\text{–a.s.},\ \forall k\in\{1,\ldots,T\}.$$

Define

$$\mathcal{A}_2 := \big\{\Delta\in\mathcal{A}(\mathbb{R}) : X_t^{0,\Delta}\in\mathbb{L}^2(\mathbb{R},\mathcal{F}_t,\mathbb{P}),\ t\in\{1,\ldots,T\}\big\}.$$

Fix some $x\in\mathbb{R}$ and some payoff $\xi\in\mathbb{L}^2(\mathbb{R},\mathcal{F}_T,\mathbb{P})$. The mean–variance hedging (MVH) is the problem of approximating, with minimal mean-squared error, a given payoff by the final value of a self-financing trading strategy in a financial market. We thus consider the problem

$$V_0(x) := \inf_{\Delta\in\mathcal{A}_2} \mathbb{E}^{\mathbb{P}}\big[\big(\xi - X_T^{x,\Delta}\big)^2\big]. \tag{0.1}$$

The goal of this exercise is to construct a candidate for the optimal strategy using the MOP.

For $\Delta\in\mathcal{A}_2$, and any $t\in\{0,\ldots,T-1\}$, we set

$$\mathcal{A}_2(t,\Delta) := \big\{\Delta'\in\mathcal{A}_2 : \Delta'_j = \Delta_j,\ \text{for } j\in\{0,\ldots,t\}\big\}.$$

Moreover, for any $v_t\in\mathbb{L}^2(\mathbb{R},\mathcal{F}_t,\mathbb{P})$, we define

$$\Gamma_t(v_t,\Delta) := \mathbb{E}^{\mathbb{P}}\Bigg[\bigg(\xi - v_t - \sum_{j=t}^{T-1}\Delta_j(S_{j+1}-S_j)\bigg)^2\Bigg|\mathcal{F}_t\Bigg],\ V_t(v_t) := \operatorname*{essinf}_{\Delta'\in\mathcal{A}_2(t,0)} \Gamma_t(v_t,\Delta').$$

1) Show that for each $t \in \{0, \dots, T-1\}$ and each $v_t \in \mathbb{L}^2(\mathbb{R}, \mathcal{F}_t, \mathbb{P})$, the collection of random variables

$$\Lambda_t(v_t) := \{\Gamma_t(v_t, \Delta') : \Delta' \in \mathcal{A}_2(t,0)\},$$

is closed under taking minima.

Let Δ^1, and Δ^2 belong to $\mathcal{A}_2(t,0)$. Define

$$\Delta^3 := \Delta^1 \mathbf{1}_A + \Delta^2 \mathbf{1}_{\Omega \setminus A},$$

where $A := \{\Gamma_t(v_t, \Delta^1) \leq \Gamma_t(v_t, \Delta^2)\}$. We have to show that $\Delta^3 \in \mathcal{A}_2(t,0)$ and

$$\Gamma_t(v_t, \Delta^3) = \min\{\Gamma_t(v_t, \Delta^1), \Gamma_t(v_t, \Delta^2)\}.$$

Since $\Delta^1 \in \mathcal{A}_2(t,0)$ and $\Delta^2 \in \mathcal{A}_2(t,0)$ we clearly have $\Delta^3_k = 0$ for $k \in \{0, \dots, t\}$. Moreover, using that $X_t^{0,\Delta^i} \in \mathbb{L}^2(\mathbb{R}, \mathcal{F}_t, \mathbb{P})$ for $i \in \{1,2\}$, and

$$X_t^{0,\Delta^3} = \mathbf{1}_A X_t^{0,\Delta^1} + \mathbf{1}_{\Omega\setminus A} X_t^{0,\Delta^2},$$

we have $X_t^{0,\Delta^3} \in \mathbb{L}^2(\mathbb{R}, \mathcal{F}_t, \mathbb{P})$ for each $t \in \{0, \dots, T\}$. This shows that $\Delta^3 \in \mathcal{A}_2(t,0)$. Further note that

$$\xi - v_t - \sum_{k=t}^{T-1} \Delta^3_k (S_{k+1} - S_k) = \mathbf{1}_A \bigg(\xi - v_t - \sum_{k=t}^{T-1} \Delta^1_k (S_{k+1} - S_k) \bigg) + \mathbf{1}_{\Omega\setminus A} \bigg(\xi - v_t - \sum_{k=t}^{T-1} \Delta^2_k (S_{k+1} - S_k) \bigg),$$

is also in $\mathbb{L}^2(\mathbb{R}, \mathcal{F}_T, \mathbb{P})$ for each $t \in \{0, \dots, T\}$, and hence $\Gamma_t(v_t, \Delta^3)$ is well-defined.

Finally, since $A \in \mathcal{F}_t$, we obtain

$$\begin{aligned}
\Gamma_t(v_t, \Delta^3) = \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - v_t - \sum_{k=t}^{T-1} \Delta^3_k (S_{k+1} - S_k) \bigg)^2 \bigg| \mathcal{F}_t \bigg] &= \mathbf{1}_A \Gamma_t(v_t, \Delta^1) + \mathbf{1}_{\Omega\setminus A} \Gamma_t(v_t, \Delta^2) \\
&= \min\{\Gamma_t(v_t, \Delta^1), \Gamma_t(v_t, \Delta^2)\}.
\end{aligned}$$

2) Show that for fixed $\Delta \in \mathcal{A}_2$, $x \in \mathbb{R}$, the process $\big(V_t(X_t^{x,\Delta})\big)_{t \in \{0,\dots,T\}}$ is an $(\mathbb{F}, \mathbb{P})$–sub-martingale.

Fix $t \in \{0, \dots, T\}$, and $k \in \{0, \dots, t\}$. We apply 1) with $v_t := X_t^{x,\Delta} \in \mathbb{L}^2(\mathbb{R}, \mathcal{F}_t, \mathbb{P})$, which, using results recalled in the Lecture Notes implies

$$\begin{aligned}
V_t\big(X_t^{x,\Delta}\big) &= \operatorname*{essinf}_{\Delta' \in \mathcal{A}_2(t,0)} \Gamma_t\big(X_t^{x,\Delta}, \Delta'\big) \\
&= \operatorname*{essinf}_{\Delta' \in \mathcal{A}_2(t,0)} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - x - \sum_{j=0}^{t-1} \Delta_j (S_{j+1} - S_j) - \sum_{j=t}^{T-1} \Delta'_j (S_{j+1} - S_j) \bigg)^2 \bigg| \mathcal{F}_t \bigg] \\
&= \lim_{n\to\infty} \downarrow \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - x - \sum_{j=0}^{t-1} \Delta_j (S_{j+1} - S_j) - \sum_{j=t}^{T-1} \Delta^n_j (S_{j+1} - S_j) \bigg)^2 \bigg| \mathcal{F}_t \bigg],
\end{aligned}$$

for a sequence $(\Delta^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_2(t,0) \subseteq \mathcal{A}_2(k,0)$. Note that $\Gamma_t(X_t^{x,\Delta}, \Delta^n)$ is in $\mathbb{L}^1(\mathbb{R}, \mathcal{F}_t, \mathbb{P})$ due to the definitions of Δ, and $(\Delta^n)_{n\in\mathbb{N}}$. Then using monotone convergence, the tower property and $(\Delta^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_2(t,0) \subseteq \mathcal{A}_2(k,0)$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[V_t(X_t^{x,\Delta}) \big| \mathcal{F}_k\big] &= \mathbb{E}^{\mathbb{P}}\bigg[\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - x - \sum_{j=0}^{t-1} \Delta_j (S_{j+1} - S_j) - \sum_{j=t}^{T-1} \Delta^n_j (S_{j+1} - S_j) \bigg)^2 \bigg| \mathcal{F}_t \bigg] \bigg| \mathcal{F}_k \bigg] \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - x - \sum_{j=0}^{t-1} \Delta_j (S_{j+1} - S_j) - \sum_{j=t}^{T-1} \Delta^n_j (S_{j+1} - S_j) \bigg)^2 \bigg| \mathcal{F}_k \bigg] \\
&\geq \operatorname*{essinf}_{\Delta' \in \mathcal{A}_2(k,0)} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\xi - x - \sum_{j=0}^{k-1} \Delta_j (S_{j+1} - S_j) - \sum_{j=k}^{T-1} \Delta'_j (S_{j+1} - S_j) \bigg)^2 \bigg| \mathcal{F}_k \bigg] \\
&= V_k\big(X_k^{x,\Delta}\big),
\end{aligned}$$

and so we have the sub-martingale property. The integrability then follows from

$$V_T(X_T^{x,\Delta}) = (\xi - X_T^{x,\Delta})^2 \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}_T, \mathbb{P}).$$

3) Show that $\Delta^\star \in \mathcal{A}_2$ is optimal if and only if the process $\big(V_t(X_t^{x,\Delta^\star})\big)_{t\in\{0,\dots,T\}}$ is an $(\mathbb{F},\mathbb{P})$-martingale.

'$\Longrightarrow$': let $\Delta^\star \in \mathcal{A}_2$ be optimal. We already know that $\big(V_t(X_t^{x,\Delta^\star})\big)_{t\in\{0,\dots,T\}}$ is a sub-martingale. To show that it is a martingale, we thus only need to show that

$$\mathbb{E}^{\mathbb{P}}\big[V_T(X_T^{x,\Delta^\star})\big] = \mathbb{E}^{\mathbb{P}}[V_0(x)].$$

By the optimality of $\Delta^\star$, we have as in the Lecture Notes (notice that $\mathcal{A}_2(0,\Delta)$ is independent of $\Delta \in \mathcal{A}_2$, and equal to $\mathcal{A}_2$)

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[V_0(x)] &= \mathbb{E}^{\mathbb{P}}\bigg[\operatorname*{essinf}_{\Delta\in\mathcal{A}_2(0)} \mathbb{E}^{\mathbb{P}}\big[(\xi - X_T^{x,\Delta})^2\big|\mathcal{F}_0\big]\bigg] \\ &= \inf_{\vartheta\in\mathcal{A}_2} \mathbb{E}^{\mathbb{P}}\big[(\xi - X_T^{x,\Delta})^2\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[(\xi - X_T^{x,\Delta^\star})^2\big] = \mathbb{E}^{\mathbb{P}}\big[V_T(X_T^{x,\Delta^\star})\big].\end{aligned}$$

This gives the desired equality.

'$\Longleftarrow$': suppose that $\big(V_t(X_t^{x,\Delta^\star})\big)_{t\in\{0,\dots,T\}}$ is a martingale. Then using $V_T\big(X_T^{x,\Delta^\star}\big) = \big(\xi - X_T^{x,\Delta^\star}\big)^2$ gives

$$\mathbb{E}^{\mathbb{P}}[V_0(x)] = \mathbb{E}^{\mathbb{P}}\big[V_T(X_T^{x,\Delta^\star})\big] = \mathbb{E}^{\mathbb{P}}\big[(\xi - X_T^{x,\Delta^\star})^2\big].$$

Moreover, the same argument as above shows that

$$\mathbb{E}^{\mathbb{P}}[V_0(x)] = \inf_{\Delta\in\mathcal{A}_2} \mathbb{E}^{\mathbb{P}}\big[(\xi - X_T^{x,\Delta})^2\big],$$

which implies that $\Delta^\star$ is optimal.

4) Show that the following recursion holds

$$V_{t-1}(x) = \operatorname*{essinf}_{\Delta'\in\mathcal{A}_2(t-1,0)} \mathbb{E}^{\mathbb{P}}\Big[V_t\big(x + \Delta'_{k-1}(S_t - S_{t-1})\big)\Big|\mathcal{F}_{t-1}\Big],\ \mathbb{P}\text{–a.s.},\ t \in \{1,\dots,T\},\ \text{with } V_T(x) = (\xi - x)^2.$$

By part the previous questions, we have for every fixed $\Delta' \in \mathcal{A}_2(t-1,0)$ that the process $V_\cdot(X_\cdot^{x,\Delta'})$ is a sub-martingale. Hence, we get

$$V_{t-1}(x) = V_{t-1}\big(X_{t-1}^{x,\Delta'}\big) \leq \mathbb{E}^{\mathbb{P}}\big[V_t(X_t^{x,\Delta'})\big|\mathcal{F}_{t-1}\big] = \mathbb{E}^{\mathbb{P}}\big[V_t\big(x + \Delta'_{t-1}(S_t - S_{t-1})\big)\big|\mathcal{F}_{t-1}\big].$$

Taking the essinf on both sides leads to

$$V_{t-1}(x) \leq \operatorname*{essinf}_{\Delta'\in\mathcal{A}_2(t-1,0)} \mathbb{E}^{\mathbb{P}}\big[V_t\big(x + \Delta'_{t-1}(S_t - S_{t-1})\big)\big|\mathcal{F}_{t-1}\big].$$

To show the converse inequality, we fix $\Delta \in \mathcal{A}_2(t-1,0)$ and then compute

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[V_t(X_t^{x,\Delta})\big|\mathcal{F}_{t-1}\big] &\leq \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\left(\xi - \big(x + \Delta_{t-1}(S_t - S_{t-1})\big) - \sum_{j=t}^{T-1} \Delta_j(S_{j+1} - S_j)\right)^2\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_{t-1}\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\xi - x - \sum_{j=t-1}^{T-1} \Delta_j(S_{j+1} - S_j)\right)^2\middle|\mathcal{F}_{t-1}\right],\end{aligned}$$

where the inequality is obtained by observing that the strategy given by $\tilde{\Delta}_j := 0$ for $j \in \{0, \dots, t-1\}$ and $\tilde{\Delta}_j := \Delta_j$ is in $\mathcal{A}_2(t-1, 0)$. Taking the essinf on both sides leads to

$$\underset{\Delta' \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \mathbb{E}^{\mathbb{P}}\big[V_t(X_t^{x,\Delta})\big|\mathcal{F}_{t-1}\big] \leq \underset{\Delta' \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \mathbb{E}^{\mathbb{P}}\Bigg[\Bigg(\xi - x - \sum_{j=t-1}^{T-1} \Delta_j(S_{j+1} - S_j)\Bigg)^2\Bigg|\mathcal{F}_{t-1}\Bigg] = V_{t-1}(x).$$

Finally $V_T(x) = (\xi - x)^2$ is clear by definition.

5) Prove by backward induction that for any $t \in \{0, \dots, T\}$

$$V_t(x) = A_t x^2 + 2B_t x + C_t,$$

where A_t, B_t, C_t are $\mathcal{F}_t$-measurable random variables, with $0 \leq A_t \leq 1$ and $A_T = 1$, $B_T = -\xi$, $C_T = \xi^2$.

For $t = T$, we have $V_T(x) = (\xi - x)^2 = x^2 - 2\xi x + \xi^2$. So $A_T = 1$, $B_T = -\xi$, and $C_T = \xi^2$, as announced.

Induction step: **suppose that $V_t(x) = A_t x^2 + 2B_t x + C_t$ with $0 \leq A_t \leq 1$. By the previous question, we need to compute**

$$\begin{aligned}
V_{t-1}(x) &= \underset{\Delta \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \mathbb{E}^{\mathbb{P}}\Big[V_t\big(x + \Delta'_{k-1}(S_t - S_{t-1})\big)\Big|\mathcal{F}_{t-1}\Big] \\
&= \underset{\Delta \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \Big\{\mathbb{E}^{\mathbb{P}}\Big[A_t\big(x + \Delta_{t-1}(S_t - S_{t-1})\big)^2 + 2B_t\big(x + \Delta_{t-1}(S_t - S_{t-1})\big) + C_t\Big|\mathcal{F}_{t-1}\Big]\Big\} \\
&= \underset{\Delta \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \Big\{\mathbb{E}^{\mathbb{P}}\big[A_t x^2 + 2B_t x + C_t\big|\mathcal{F}_{t-1}\big] + 2\Delta_{t-1}\mathbb{E}^{\mathbb{P}}\big[xA_t(S_t - S_{t-1}) + B_t(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] \\
&\qquad + \Delta_{t-1}^2\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big]\Big\}.
\end{aligned}$$

This amounts to the optimisation over Δ_{t-1} of a quadratic polynomial, and therefore depends on whether the leading coefficient is 0 or not.

On the event $G_{t-1} := \big\{\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big] = 0\big\}$, we first observe by Cauchy–Schwarz's inequality for conditional expectations that

$$\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big]^2 = \mathbb{E}^{\mathbb{P}}\Big[\sqrt{A_t}\sqrt{A_t}(S_t - S_{t-1})\Big|\mathcal{F}_{t-1}\Big]^2 \leq \mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}]\mathbb{E}^{\mathbb{P}}\big[A_t S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big] = 0.$$

On the other hand, note that $B_t^2 \leq A_t C_t$ because $V_t(x) \geq 0$. This implies $\{A_t = 0\} \subseteq \{B_t = 0\}$, and thus

$$\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\mathbf{1}_{G_{t-1}}\big] = \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big]\mathbf{1}_{G_{t-1}}\Big] = 0.$$

Using $A_t(S_t - S_{t-1})^2\mathbf{1}_{G_{t-1}} \geq 0$, $\mathbb{P}$–a.s., we obtain $A_t(S_t - S_{t-1})^2\mathbf{1}_{G_{t-1}} = 0$, $\mathbb{P}$–a.s. Thus $B_t(S_t - S_{t-1})^2\mathbf{1}_{G_{t-1}} = 0$, $\mathbb{P}$–a.s., and hence $B_t(S_t - S_{t-1})\mathbf{1}_{G_{t-1}} = 0$, $\mathbb{P}$–a.s. This yields $\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]\mathbf{1}_{G_{t-1}} = 0$, $\mathbb{P}$–a.s. To sum up, we have obtained the implication

$$\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big] = 0 \Longrightarrow \mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] = 0, \text{ and } \mathbb{E}^{\mathbb{P}}\big[B_t(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] = 0.$$

Now the optimisation problem on G_{t-1} thus becomes

$$V_{t-1}(x) = \underset{\Delta \in \mathcal{A}_2(t-1,0)}{\operatorname{essinf}} \mathbb{E}^{\mathbb{P}}\big[A_t x^2 + 2B_t x + C_t\big|\mathcal{F}_{t-1}\big] = \mathbb{E}^{\mathbb{P}}\big[A_t x^2 + 2B_t x + C_t\big|\mathcal{F}_{t-1}\big].$$

Thus $V_{t-1}(x) = A_{t-1}x^2 + 2B_{t-1}x + C_{t-1}$, with $A_{t-1} = \mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}]$, $B_{t-1} = \mathbb{E}^{\mathbb{P}}[B_t|\mathcal{F}_{t-1}]$, $C_{t-1} = \mathbb{E}^{\mathbb{P}}[C_t|\mathcal{F}_{t-1}]$. This yields $0 \leq A_{t-1} \leq 1$, and verifies the induction step. Moreover, since the objective functional does not depend on Δ_{t-1}, we can choose it arbitrarily.

On $\Omega \setminus G_{t-1} = \big\{\mathbb{E}^{\mathbb{P}}\big[A_t(S_t - S_{t-1})^2\big|\mathcal{F}_{t-1}\big] > 0\big\}$, the optimiser is

$$\Delta_{t-1}^\star(x) := -\frac{\mathbb{E}^{\mathbb{P}}[(xA_t + B_t)(S_t - S_{t-1})|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]}.$$

With the convention $0/0 := 0$, we ensure that $\Delta^\star_{t-1}(x)$ is well-defined on both G_{t-1} and $\Omega \setminus G_{t-1}$. Now substituting $\Delta^\star_{t-1}(x)$ in the above gives after some computations

$$\begin{aligned} V_{t-1}(x) = x^2 &\left(\mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}] - \frac{(\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})|\mathcal{F}_{t-1}])^2}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} \right) \\ &+ 2x \left(\mathbb{E}^{\mathbb{P}}[B_t|\mathcal{F}_{t-1}] - \frac{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} \right) \\ &+ \mathbb{E}^{\mathbb{P}}[C_t|\mathcal{F}_{t-1}] - \frac{(\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}])^2}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]}. \end{aligned}$$

We can thus set

$$\begin{aligned} A_{t-1} &:= \mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}] - \frac{(\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})|\mathcal{F}_{t-1}])^2}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]}, \\ B_{t-1} &:= \mathbb{E}^{\mathbb{P}}[B_t|\mathcal{F}_{t-1}] - \frac{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]}, \\ C_{t-1} &:= \mathbb{E}^{\mathbb{P}}[C_t|\mathcal{F}_{t-1}] - \frac{(\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}])^2}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]}, \end{aligned}$$

and check that A_{t-1} takes values in $(0,1)$ by Cauchy–Schwarz's inequality. This ends the proof.

6) Use the DPP to construct a candidate for an optimal strategy $\Delta^\star$.

Note that by the DPP and what precedes, we have

$$\begin{aligned} V_{t-1}(v_{t-1}) &= \operatorname*{essinf}_{\Delta \in \mathcal{A}_2(t-1,0)} \mathbb{E}^{\mathbb{P}}\big[V_t(v_{t-1} + \Delta_{t-1}(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] \\ &= \operatorname*{essinf}_{\Delta \in \mathcal{A}_2(t-1,0)} \mathbb{E}^{\mathbb{P}}\Big[A_t\big(v_{t-1} + \Delta_{t-1}(S_t - S_{t-1})\big)^2 + 2B_t\big(v_{t-1} + \Delta_{t-1}(S_t - S_{t-1})\big) + C_t\Big|\mathcal{F}_{t-1}\Big]. \end{aligned}$$

Setting the differential w.r.t Δ_{t-1} to 0, we see that the first order condition is

$$2\mathbb{E}^{\mathbb{P}}\big[A_t\big(v_{t-1} + \Delta_{t-1}(S_t - S_{t-1})\big)(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] + 2\mathbb{E}^{\mathbb{P}}\big[B_t(S_t - S_{t-1})\big|\mathcal{F}_{t-1}\big] = 0.$$

Using the measurability of A_t, B_t, and C_t, we get that the optimal $\Delta^\star_{t-1}(v_{t-1})$ for a given v_{t-1} is

$$\Delta^\star_{t-1}(v_{t-1}) := -\frac{\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} - \frac{\mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} v_{t-1}.$$

We can thus proceed recursively and have

$$\Delta^\star_t := -\frac{\mathbb{E}^{\mathbb{P}}[B_t(S_t - S_{t-1})|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} - \frac{\mathbb{E}^{\mathbb{P}}[A_t|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{P}}[A_t(S_t - S_{t-1})^2|\mathcal{F}_{t-1}]} X^{x,\Delta^\star}_{t-1}.$$

Thus $\Delta^\star$ give a candidate for an optimal strategy.