Introduction to Mathematical Finance
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Assignment 12 - solutions

On utility maximisation

Consider a general arbitrage-free single-period market, where the interest rate is taken to be 0. Fix $x > 0$ and let $U : [0, \infty) \longrightarrow \mathbb{R}$ be a concave, increasing (utility) function, continuously differentiable on $(0, \infty)$, such that

$$\sup_{\xi \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\big[U\big(x + \xi \cdot (S_1 - S_0)\big)\big] < \infty, \tag{0.1}$$

with

$$\mathcal{A}(x) := \big\{\xi \in \mathbb{R}^d : x + \xi \cdot (S_1 - S_0) \geq 0,\ \mathbb{P}\text{–a.s.}\big\}.$$

Furthermore, assume that the supremum is attained in an interior point $\xi^\star$ of $\mathcal{A}(x)$.

1) Show that

$$U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)|S_1 - S_0| \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P}),$$

and the *first order condition*

$$\mathbb{E}^{\mathbb{P}}\big[U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)(S_1 - S_0)\big] = 0.$$

Let η be any non-zero vector. Then, by the assumption that $\xi^\star$ is an interior point of $\mathcal{A}(x)$, we have

$$x + \xi^\star \cdot (S_1 - S_0) > 0,\ \mathbb{P}\text{–a.s.},$$

so that for any $\varepsilon > 0$ small enough, we will have

$$x + (\xi^\star + \varepsilon\eta) \cdot (S_1 - S_0) \geq 0,\ \mathbb{P}\text{–a.s.},$$

and thus $\xi^\star + \varepsilon\eta \in \mathcal{A}(x)$. Define then for ε small enough

$$\Delta_\varepsilon^\eta := \frac{U\big(x + (\xi^\star + \varepsilon\eta) \cdot (S_1 - S_0)\big) - U\big(x + \xi^\star \cdot (S_1 - S_0)\big)}{\varepsilon}.$$

On the set $\big\{\eta \cdot (S_1 - S_0) = 0\big\}$, Δ_ε^η is equal to 0, and on the set $\big\{\eta \cdot (S_1 - S_0) \neq 0\big\}$

$$\Delta_\varepsilon^\eta = \eta \cdot (S_1 - S_0)\frac{U\big(x + (\xi^\star + \varepsilon\eta) \cdot (S_1 - S_0)\big) - U\big(x + \xi^\star \cdot (S_1 - S_0)\big)}{\varepsilon\eta \cdot (S_1 - S_0)},$$

so Δ_ε^η is monotonically[1] decreasing to $\eta \cdot (S_1 - S_0)U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)$ as $\varepsilon \searrow 0$. Note that by concavity of U, $U'(0)$ is well-defined as the right-derivative of U at 0.

From (0.1) we know that $U\big(x + \xi \cdot (S_1 - S_0)\big) \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P})$ for all $\xi \in \mathcal{A}(x)$, and so $\Delta_\varepsilon^\eta \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P})$ as well. Next note that $\varepsilon \longmapsto \Delta_\varepsilon^\eta$ is decreasing. Hence for $\varepsilon \searrow 0$, $\Delta_\varepsilon^\eta \nearrow$ and we can use monotone convergence to deduce

$$-\infty < \mathbb{E}^{\mathbb{P}}[\Delta_\varepsilon^\eta] \leq \mathbb{E}^{\mathbb{P}}\big[U'(x + \xi^\star \cdot (S_1 - S_0))\eta \cdot (S_1 - S_0)\big] = \lim_{\varepsilon \searrow 0} \mathbb{E}^{\mathbb{P}}[\Delta_\varepsilon^\eta] \leq 0.$$

Therefore, $U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)\eta \cdot (S_1 - S_0) \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P})$.

Finally, since η can be chosen arbitrarily, $U'\big(x + \xi^\star \cdot (S_1 - S_0)\big)(S_1 - S_0) \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P})$ and

$$\eta \cdot \mathbb{E}^{\mathbb{P}}\big[U'(x + \xi^\star \cdot (S_1 - S_0))(S_1 - S_0)\big] \leq 0,$$

with $\eta := \mathbb{E}^{\mathbb{P}}\big[U'(x + \xi^\star \cdot (S_1 - S_0))(S_1 - S_0)\big]$ implies

$$\mathbb{E}^{\mathbb{P}}\big[U'(x + \xi^\star \cdot (S_1 - S_0))(S_1 - S_0)\big] = 0.$$

[1] This is easily seen by splitting into two cases depending on the sign of $\eta \cdot (S_1 - S_0)$.

2) Show that $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{U'(x+\xi^\star\cdot(S_1-S_0))}{\mathbb{E}^{\mathbb{P}}[U'(x+\xi^\star\cdot(S_1-S_0))]},$$

is a risk-neutral measure.

By $1)$**,** $\mathbb{Q}$ **satisfies the martingale property, and is equivalent to** $\mathbb{P}$ **since** $U'>0$**. The only thing to check is that** $\mathbb{Q}$ **is well-defined, that is to say that** $U'(x+\xi^\star\cdot(S_1-S_0))\in\mathbb{L}^1(\mathbb{R},\mathcal{F},\mathbb{P})$**. Observe that**

$$U'(x+\xi^\star\cdot(S_1-S_0)) = U'(x+\xi^\star\cdot(S_1-S_0))\mathbf{1}_{\{\xi^\star\cdot(S_1-S_0)\leq -x/2\}} + U'(x+\xi^\star\cdot(S_1-S_0))\mathbf{1}_{\{\xi^\star\cdot(S_1-S_0)\geq -x/2\}}.$$

The second term is bounded by $U'(x/2)$ **since** U' **is non-increasing. Again using** $1)$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[U'(x+\xi^\star\cdot(S_1-S_0))\mathbf{1}_{\{\xi^\star\cdot(S_1-S_0)\leq -x/2\}}\Big] &\leq \mathbb{E}^{\mathbb{P}}\bigg[\frac{-\xi^\star\cdot(S_1-S_0)}{x/2}U'(x+\xi^\star\cdot(S_1-S_0))\mathbf{1}_{\{\xi^\star\cdot(S_1-S_0)\leq -x/2\}}\bigg]\\ &\leq \frac{2}{x}\mathbb{E}^{\mathbb{P}}\Big[\big|\xi^\star\cdot(S_1-S_0)\big|U'\big(x+\xi^\star\cdot(S_1-S_0)\big)\Big]<\infty,\end{aligned}$$

which ends the proof.

Why do we need super-martingale deflators?

We put ourselves in the setting of the duality approach to utility maximisation. The goal of this exercise is to illustrate why we need, for fixed $z>0$, to work with the larger set $\mathcal{Z}(z)$ in the dual problem $j(z)$, instead of the set $z\mathcal{M}(S)$ of densities of risk-neutral measures (multiplied by z).

We construct a one-period market defined on a *countable* probability space Ω. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers such that

$$\sum_{n=0}^{\infty} p_n = 1, \text{ and } \lim_{n\to+\infty} p_n = 0.$$

We also let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers starting at $x_0=2$ and also decreasing to 0 and n goes to $+\infty$, but less fast than $(p_n)_{n\in\mathbb{N}}$, in a sense to be made precise later on.

Finally, we consider a market with constant non-risky asset and one risky asset, starting from $S_0=1$, and such that the $\mathbb{P}$-distribution of S_1 is given by

$$\mathbb{P}[S_1=x_n]=p_n,\ n\in\mathbb{N}.$$

We equip the probability space with the natural filtration of S.

1) Show that the market is arbitrage free, and argue that $\mathcal{M}(S)\neq\emptyset$. Is the market complete?

Let $\Delta\in\mathbb{R}$ **be an arbitrage opportunity. Then we must have**

$$X_1^{0,\Delta}=\Delta(S_1-S_0)=\Delta(S_1-1)\geq 0,\ \mathbb{P}\text{–a.s., } \textbf{and } \mathbb{P}\big[\Delta(S_1-1)>0\big]>0.$$

If $\Delta<0$**, then we must have** $S_1\leq 1$**,** P**–a.s., which is not possible since** $x_0=2$**, and** $p_0>0$**. Similarly, for** $\Delta>0$**, the arbitrage condition would imply** $S_1\geq 1$**,** $\mathbb{P}$**–a.s., which again is not possible since** $x_n\to 0$ **as** $n\to\infty$**, and all the** $(p_n)_{n\in\mathbb{N}}$ **are positive. The only possibility is to have thus** $\Delta=0$**, in which case** Δ **cannot be an arbitrage opportunity. The first FTAP then ensures that** $\mathcal{M}(S)\neq\emptyset$**.**

The market is not complete however. Indeed, given a contingent claim with payoff ξ**, in order to replicate it one has to solve the system of equations** $\xi=x+\Delta(S_1-1)$**. This is a system with infinitely many equations, but only two unknowns. Hence for general** ξ**, the system does not admit a solution. For example, the claim** $\xi=(S_1-1)^2$ **is not replicable.**

2) Determine an interval $[a,b] \subset \mathbb{R}$ such that $\mathcal{V}(1) = [a,b]$, that is to say $X_1^{1,\Delta} \geq 0$, $\mathbb{P}$–almost surely, if and only if $\Delta \in [a,b]$.

The condition $X_1^{1,\Delta} = 1 + \Delta(S_1 - 1) \geq 0$, $\mathbb{P}$–a.s. is equivalent to the conditions

$$1 + \Delta(x_n - 1) \geq 0,\ n \in \mathbb{N}.$$

With $n = 0$, we get $\Delta \geq -1$, and letting n go to $+\infty$, we get $\Delta \leq 1$. Now conversely, since $(x_n)_{n\in\mathbb{N}}$ takes values in $[0,2]$, whenever $\Delta \in [-1,1]$, we have for any $n \in \mathbb{N}$ such that $x_n \geq 1$

$$1 + \Delta(x_n - 1) \geq 2 - x_n \geq 0,$$

and for any $n \in \mathbb{N}$ such that $x_n < 1$

$$1 + \Delta(x_n - 1) \geq x_n \geq 0.$$

Hence $a = -1$ and $b = 1$.

3) Assume for this question and all the remaining ones that the following series (whose terms are negative for n large enough) are well-defined

$$\sum_{n\in\mathbb{N}} p_n \log(x_n),\ \sum_{n\in\mathbb{N}} p_n \frac{x_n - 1}{x_n},$$

and that

$$\sum_{n\in\mathbb{N}} p_n \frac{x_n - 1}{x_n} > 0.$$

Maximise the function

$$f(\Delta) := \mathbb{E}^{\mathbb{P}}\big[\log(X_1^{1,\Delta})\big],$$

over $[a,b]$. Derive the optimal investment $\Delta^\star \in \mathcal{V}(1)$.

We have first

$$f(\Delta) = \sum_{n\in\mathbb{N}} p_n \log\big(1 + \Delta(x_n - 1)\big),$$

where we still need to ensure that the series is well-defined. Notice first that for any $\Delta \in (-1,1)$, and for any $n \in \mathbb{N}$, we have

$$\log\big(1 - |\Delta|\big) \leq \log\big(1 + \Delta(x_n - 1)\big) \leq \log\big(1 + |\Delta|\big).$$

This ensures that for any $\Delta \in (-1,1)$, the series $\sum_{n\in\mathbb{N}} p_n \log\big(1 + \Delta(x_n - 1)\big)$ is absolutely convergent. In addition, we have, still for any $\Delta \in (-1,1)$

$$-\frac{p_n}{1-\Delta} \leq \frac{\mathrm{d}}{\mathrm{d}\Delta}\Big(p_n \log\big(1 + \Delta(x_n - 1)\big)\Big) = \frac{p_n(x_n - 1)}{1 + \Delta(x_n - 1)} \leq \frac{p_n}{1+\Delta}.$$

Hence for any $\varepsilon > 0$ small enough, and any $\Delta \in (-1+\varepsilon, 1-\varepsilon)$, we can control $\log\big(1 + \Delta(x_n - 1)\big)$ and $\frac{\mathrm{d}}{\mathrm{d}\Delta}\Big(p_n \log\big(1 + \Delta(x_n - 1)\big)\Big)$, uniformly in Δ, by the terms of absolutely convergent series. This ensures that f is not only well-defined on $(-1,1)$, but also C^1 on $(-1,1)$.

The additional assumptions in this question also ensure that f and f' are both also well-defined at 1, and continuous there. Moreover, notice that the map $\Delta \longmapsto \frac{p_n(x_n-1)}{1+\Delta(x_n-1)}$ is non-increasing, so that for any $\Delta \in (-1,1]$, we have $f'(\Delta) \geq f'(1) > 0$, by assumption. This shows that f attains its maximum over $(-1,1]$ at 1, and since $\mathcal{V}(1) = [-1,1]$, we actually found that $\Delta^\star = 1$.

4) Compute explicitly (in terms of $(x_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$) the value function $v(x)$ for logarithmic utility. Show that $v'(1) = 1$.

We recall that

$$v(x) = \sup_{\Delta\in\mathcal{V}(x)} \mathbb{E}^{\mathbb{P}}\big[\log(X_1^{x,\Delta})\big] = \sup_{\Delta\in\mathcal{V}(x)} \mathbb{E}^{\mathbb{P}}\big[\log(xX_1^{1,\Delta})\big],$$

which by the previous question is maximised for $\Delta^\star = 1$. Hence

$$v(x) = \mathbb{E}^{\mathbb{P}}\big[\log(xX_1^{1,1})\big] = \sum_{n\in\mathbb{N}} p_n \log(xx_n) = \log(x) + \sum_{n\in\mathbb{N}} p_n \log(x_n),$$

from which it is immediate that $v'(1) = 1$.

5) Compute the corresponding dual optimiser $Z^\star \in \mathcal{Z}(1)$.

By the Lecture notes (all the hypotheses necessary here are satisfied, as you can check directly), for any $x > 0$ and $z > 0$, the solution $h_x^\star$ of the primal problem

$$v(x) = \sup_{\Delta\in\mathcal{V}(x)} \mathbb{E}^{\mathbb{P}}[U(X_1^{x,\Delta})] = \sup_{h\in\mathcal{C}(x)} E[U(h)],$$

and the solution $\xi_z^\star$ of the dual problem

$$j(z) = \inf_{Z\in\mathcal{Z}(z)} \mathbb{E}^{\mathbb{P}}[J(Z_1)] = \inf_{\xi\in\mathcal{D}(z)} \mathbb{E}^{\mathbb{P}}[J(\xi)],$$

are related by

$$h_x^\star = I(\xi_{z_x}^\star),$$

where $I = (U')^{-1}$, and $z_x > 0$ is given by the relation $x = -j'(z_x)$.

Here we have shown that $h_x^\star = X_1^{x,1} = x - 1 + S_1$ for any $x > 0$. Besides, $I(y) = y^{-1}$. Hence, taking x so that $z_x = 1$ (that is to say taking $x = -j'(1)$, we get

$$\xi_1^\star = \frac{1}{-j'(1) - 1 + S_1}.$$

Now recall that v and j are convex conjugate, which in particular shows that $j'(1) = -(v')^{(-1)}(1) = -1$ by the previous question. Hence

$$\xi_1^\star = \frac{1}{S_1}.$$

6) Assume now that

$$\sum_{n\in\mathbb{N}} \frac{p_n}{x_n} < 1.$$

Conclude that $Z^\star \in \mathcal{Z}(1)$ is not a martingale, but only a super-martingale. In particular, $Z^\star$ is not the density process of a martingale measure for the process S, and hence the infimum

$$\inf_{\mathbb{Q}\in\mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}\left[J\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right],$$

is not attained.

From the previous question, the optimal process $Z^\star \in \mathcal{Z}(1)$ is such that $Z_0^\star = 1$ and $Z_1^\star = S_1^{-1}$. Next, we have

$$\mathbb{E}^{\mathbb{P}}[Z_1^\star] = \sum_{n\in\mathbb{N}} \frac{p_n}{x_n} < 1 = Z_0^\star,$$

proving thus that $Z^\star$ cannot be a super-martingale.

7) Provide an example of $(x_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ such that the following series are well-defined

$$\sum_{n\in\mathbb{N}} p_n \log(x_n),\ \sum_{n\in\mathbb{N}} p_n \frac{x_n - 1}{x_n},$$

and such that

$$\sum_{n\in\mathbb{N}} p_n \frac{x_n - 1}{x_n} > 0,\ \sum_{n\in\mathbb{N}} \frac{p_n}{x_n} < 1.$$

One can take for instance

$$p_0 := 1 - \alpha,\ p_n := \frac{\alpha}{2^n},\ x_n := \frac{1}{n},\ n \in \mathbb{N} \setminus \{0\},$$

where $\alpha > 0$ is chosen small enough so that

$$\frac{1-\alpha}{2} - \alpha \sum_{n=1}^{+\infty} \frac{n-1}{2^n} = \frac{1-\alpha}{2} - \alpha = 1 - \frac{3\alpha}{2} > 0,$$

that is to say $\alpha < 2/3$.