Introduction to Mathematical Finance
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Assignment 2

Cross–hedging

When one wishes to cover an existing position on a given asset by using a forward contract, one ensures that the corresponding asset is as close as possible to the original one. In the case where these two assets are not perfectly correlated, we will see how to compute the optimal quantity of forward contracts to detain.

Let then t be the current date. An investor wishes to hedge a portfolio of m assets with price C_s at $s \geq t$, with maturity T. He decides to hedge by buying, at t, x forward contracts at the price f_t.

1) What is the net value V_T of the portfolio of the investor at T?

The net value of the portfolio of the investor at time T is given by the difference between what he paid to construct it and the corresponding terminal value, that is to say

$$V_T = mC_T + xf_T - \frac{mC_t + xf_t}{B(t,T)},$$

where you should recall to compound the second term, since it represents money from time t, and not time T.

2) We denote

$$\sigma_V^2 := \mathbb{V}\mathrm{ar}(V_T),\ \sigma_C^2 := \mathbb{V}\mathrm{ar}(C_T),\ \sigma_f^2 := \mathbb{V}\mathrm{ar}(f_T),\ \rho_{Cf} := \mathbb{C}\mathrm{ov}(C_T, f_T).$$

Show that in order to minimise σ_V^2, he must choose

$$x = -m\frac{\rho_{Cf}}{\sigma_f^2}.$$

Let us start by computing σ_V^2. We have, recalling that C_t, f_t and $B(t,T)$ are deterministic quantities (seen from time t)

$$\sigma_V^2 = \mathbb{V}\mathrm{ar}[V_T] = \mathbb{V}\mathrm{ar}[mC_T + xf_T] = m^2\sigma_C^2 + x^2\sigma_f^2 + 2mx\rho_{Cf}.$$

This is a strictly convex function of x which attains its minimum at its unique critical point, which verifies

$$2x\sigma_f^2 + 2m\rho_{Cf} = 0 \iff x = -m\frac{\rho_{Cf}}{\sigma_f^2}.$$

3) Show that the risk can only be canceled if $\rho_{Cf}^2 = \sigma_C^2\sigma_f^2$. What does it imply for the assets C and f?

With the above choice of x, the variance of V_T becomes

$$\sigma_V^2 = \frac{m^2}{\sigma_f^2}\left(\sigma_C^2\sigma_f^2 - \rho_{Cf}^2\right),$$

which can be equal to 0 if and only if

$$\rho_{Cf}^2 = \sigma_C^2\sigma_f^2,$$

which in turn means that C and f have to be either completely correlated or anti-correlated.

4) Application: an airline knows that that it will need to buy 2M litres of oil in one month. As there are no futures contracts on oil, the airline will use futures on gasoil. We give below the variations[1] of gasoil price ΔS and the variations of the corresponding futures prices Δf

[1] This means here that ΔS is the absolute variation of S between two consecutive months, and similarly for Δf.

Month	Δf	ΔS
1	0.021	0.029
2	0.035	0.020
3	-0.046	-0.044
4	0.001	0.008
5	0.044	0.026
6	-0.029	-0.019
7	-0.026	-0.010
8	-0.029	-0.007
9	0.048	0.043
10	-0.006	0.011
11	-0.036	-0.036
12	-0.011	-0.018
13	0.019	0.009
14	-0.027	-0.032
15	0.029	0.023

Given that each futures contract on gasoil is for a quantity of 42000 litres, how many contracts must the company buy? What is its risk?

Let us start by computing the empirical variances and covariance for S and f. We have

$$\mathbb{V}\text{ar}[\Delta f] = 0.00098241,\ \mathbb{V}\text{ar}[\Delta S] = 0.00068931,\ \mathbb{C}\text{ov}(\Delta f, \Delta S) = 0.000763971.$$

Using the same reasoning as in the previous question, we get that the desired number of contracts x is

$$x = \frac{2000000}{42000} \times \frac{0.000763971}{0.00098241} = 37.03,$$

meaning that the company should buy 37 contracts. Careful here, since we will be buying the oil in the future, m here is negative, which changes the overall sign. Moreover, the residual variance in this case is 0.2159.

Another way to think about this is to go back to prices instead of prices variations. In this case, we get, since S_0 remains unknown

Month	$f - f_0$	$S - S_0$
1	0.021	0.029
2	0.056	0.049
3	0.01	0.005
4	0.011	0.013
5	0.055	0.039
6	0.026	0.02
7	0	0.01
8	-0.029	0.003
9	0.019	0.046
10	0.013	0.057
11	-0.023	0.021
12	-0.034	0.003
13	-0.015	0.012
14	-0.042	-0.02
15	-0.013	0.003

We then have

$$\mathbb{V}\text{ar}[f - f_0] = 0.000893667,\ \mathbb{V}\text{ar}[S - S_0] = 0.00044481,\ \mathbb{C}\text{ov}(f - f_0, S - S_0) = 0.000447444.$$

We then get

$$x \approx 23.84,$$

meaning that in this case we should buy 24 **contracts. Of course, the two approaches lead to (reasonably) similar results.**

Swaps

We consider two firms A and B, which can borrow on the market at the following conditions

- Firm A : 8.5% fixed rate or Euribor +3%, for borrowing 10M for 8 years.
- Firm B : 4.5% fixed rate or Euribor +1%, for borrowing 10M for 8 years.

They both decide to go to the same bank, which will be in charge to design a swap contract between them (and which thus will take a fee). Depending on the preferences of the firms (fixed rate for A and floating for B, or fixed rate for B and floating for A), find, if they exist, all the swap contracts (including the fees for the bank) which can improve the borrowing conditions of both firms. What is the maximal fee that the bank can get?

Let us start by considering the case where A **prefers a floating rate and** B **a fixed rate, and let us ignore for now the presence of the bank.** A **thus borrows first at a fixed rate and** B **at the floating rate, and they then enter into a swap. Let us denote by** x **the interest that** B **pays to** A **in exchange for the floating rate. The final rates for each firms are**

$$\begin{aligned}&\textbf{For } A\textbf{:}\ 8.5 + \textbf{Euribor} - x,\\ &\textbf{For } B\textbf{:}\ \textbf{Euribor} + 1 + x - \textbf{Euribor} = 1 + x.\end{aligned}$$

For the swap to be profitable, we must have

$$\begin{cases} 8.5 - x \leq 3, \\ 1 + x \leq 4.5, \end{cases} \iff \begin{cases} 5.5 \leq x, \\ x \leq 3.5, \end{cases},$$

which is impossible.

In the opposite case, we deduce similarly the following final rates (y **denotes here the interest paid by** A **to** B **in exchange for Euribor**

$$\begin{aligned}&\textbf{For } A\textbf{:}\ \textbf{Euribor} + 3 - \textbf{Euribor} + y = 3 + y,\\ &\textbf{For } B\textbf{:}\ 4.5 + \textbf{Euribor} - y.\end{aligned}$$

For the swap to be profitable, we must have

$$\begin{cases} 3 + y \leq 8.5, \\ 4.5 - y \leq 1, \end{cases} \iff y \in [3.5, 5.5].$$

Swaps are therefore possible in this case only. Furthermore, the fee that the bank can get should be such that the swap still remains profitable for both A **and** B**. Hence, if the fee (expressed as interest) taken from** A **is** f_A **and the one from** B **is** f_B**, we should have**

$$f_A + f_B \leq 5.5 - 3.5 = 2.$$

Hence the highest cumulative fee that the bank can charge cannot exceed 2%

Speed of a Bond

We consider a Bond with maturity t_n delivering cash-flows $F(t_i)$ at the successive dates $(t_i)_{1 \leq i \leq n}$. We also assume that the interest rate curve is flat at the fixed level $r > 0$. Interest rates are supposed to be compounded once per year.

1) What is the formula giving the price of this Bond at time $t = 0$?

By the course, we know that the formula is

$$P = \sum_{i=1}^{n} F(t_i)B(0, t_i) = \sum_{i=1}^{n} \frac{F(t_i)}{(1+r)^{t_i}}.$$

2) Recall the definition of the Duration D and the Convexity C for a Bond, and provide first and second order approximations of the price variation ΔP due to an interest rate variation of Δr.

We have

$$D := -\frac{1+r}{P}\frac{\partial P}{\partial r}, \; C := \frac{1}{P}\frac{\partial^2 P}{\partial r^2}.$$

The first and second-order Taylor expansions are

$$\Delta P := P(r + \Delta r) - P(r) \approx \frac{\partial P}{\partial r}\Delta r = -\frac{P}{1+r}D\Delta r,$$
$$\Delta P := P(r + \Delta r) - P(r) \approx \frac{\partial P}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\Delta r)^2 = -\frac{P}{1+r}D\Delta r + P\frac{C}{2}(\Delta r)^2.$$

3) We know define the Speed κ of a Bond as follows

$$\kappa := -\frac{1}{P}\frac{\partial^3 P}{\partial r^3}.$$

Give an explicit formula for κ in this exercise, and its sign. Deduce a third order approximation of the price variation ΔP due to an interest rate variation of Δr, in terms of D, C and κ. Deduce whether we over or underestimate the variations of P when we neglect the third order term.

We have after direct computations

$$\kappa = \frac{1}{P(1+r)^3}\sum_{i=1}^{n} \frac{t_i(t_i+1)(t_i+2)F(t_i)}{(1+r)^{t_i}}.$$

Provided that the $F(t_i)$ are non-negative, we deduce that $\kappa \geq 0$. Furthermore, the third order expansion is now

$$\begin{aligned}\Delta P := P(r + \Delta r) - P(r) &\approx \frac{\partial P}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\Delta r)^2 + \frac{(\Delta r)^3}{6}\frac{\partial^3 P}{\partial r^3} \\ &= -\frac{P}{1+r}D\Delta r + P\frac{C}{2}(\Delta r)^2 - \frac{P}{6}\kappa(\Delta r)^3.\end{aligned}$$

Since the third order term above has the same sign as $-\Delta r$, this implies that when $\Delta r > 0$, we underestimate how much P diminishes (recall that P decreases with r), and when $\Delta r < 0$, we underestimate how much P increases.

3) We consider a Bond with maturity 2 years, face value of \$100, a coupon rate of 10%, and we assume $r = 8\%$ and that coupons are paid annually. Compute the value, the Duration, the Convexity and the Speed of this Bond. Compute its new exact price when r diminishes brutally to 1%, as well as the approximated new prices using the first, second, and third order approximations. Comment.

We have

$$
\begin{aligned}
P &= \frac{10}{1+8\%} + \frac{10}{(1+8\%)^2} + \frac{100}{(1+8\%)^2} \approx 103.566,\\
D &= \frac{1}{103.566}\left(\frac{1\times 10}{1+8\%} + \frac{2\times 10}{(1+8\%)^2} + \frac{2\times 100}{(1+8\%)^2}\right) \approx 1.911,\\
C &= \frac{1}{(1+0.08)^2 103.566}\left(\frac{1\times 2\times 10}{1+8\%} + \frac{2\times 3\times 10}{(1+8\%)^2} + \frac{2\times 3\times 100}{(1+8\%)^2}\right) \approx 4.837,\\
\kappa &= \frac{1}{(1+0.08)^3 103.566}\left(\frac{1\times 2\times 3\times 10}{1+8\%} + \frac{2\times 3\times 4\times 10}{(1+8\%)^2} + \frac{2\times 3\times 4\times 100}{(1+8\%)^2}\right) \approx 17.775.
\end{aligned}
$$

If r becomes 1%, the new price P' of the Bond is

$$
P = \frac{10}{1+1\%} + \frac{10}{(1+1\%)^2} + \frac{100}{(1+1\%)^2} \approx 117.734,
$$

which represents a variation $\Delta P = 117.734 - 103.566 = 14.168$. The different approximations give respectively

$$
\begin{aligned}
\Delta P &\approx \frac{103.566}{1+8\%} \times 1.911 \times (7\%) = 12.828,\\
\Delta P &\approx \frac{103.566}{1+8\%} \times 1.911 \times (7\%) + 103.566 \times \frac{4.837}{2} \times (-7\%)^2 = 14.055,\\
\Delta P &\approx \frac{103.566}{1+8\%} \times 1.911 \times (7\%) + 103.566 \times \frac{4.837}{2} \times (-7\%)^2 + 103.566 \times \frac{17.775}{6} \times (7\%)^3 = 14.160.
\end{aligned}
$$

As expected, the accuracy of the approximation increases with the order, and reaches a satisfactory value (with a precision of 10^{-2}) at the third order only.