Introduction to Mathematical Finance
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Assignment 3 (Solutions)

1. **Call option properties**

Let $C_t(T,K;S)$ be the price at some time $t \geq 0$ of an European Call option with strike $K \geq 0$, maturity $T \geq 0$ and underlying asset with value $(S_t)_{t\in[0,T]}$.

1) Prove that for any $(T_1,T_2,t,K) \in [0,+\infty) \times [T_1,+\infty) \times [0,T_1] \times [0,+\infty)$, we have

$$C_t(T_1, KB(T_1,T_2);S) \leq C_t(T_2,K;S).$$

In particular, show that the map $T \longmapsto C_t(T,K;S)$ is non–decreasing when interest rates are non–negative.

Consider the following 2 portfolios

- P_1**: long one call option with strike** $KB(T_1,T_2)$ **and maturity** T_1**.**
- P_2**: long one call option with strike** K **and maturity** T_2**.**

By Proposition 1.4.15.(i)**, we know that**

$$C_{T_1}(T_2,K;S) \geq \big(S_{T_1} - KB(T_1,T_2)\big)^+ = C_{T_1}\big(T_1, KB(T_1,T_2);S\big).$$

Hence, the value of the portfolio P_1 **at** T_1 **is always below the value of the portfolio** P_2 **at time** T_1**, which implies the desired inequality by the no–dominance principle. The final statement comes from the fact that when interest rates are non–negative, we have** $B(T_1,T_2) \in (0,1]$**, and that Call prices are non–increasing with their strike by** (ii)**.**

2) Prove that for any $(T,t,K_1,K_2) \in [0,+\infty) \times [0,T] \times [0,+\infty) \times [0,+\infty)$, we have

$$\big|C_t(T,K_1;S) - C_t(T,K_2;S)\big| \leq B(t,T)\big|K_2 - K_1\big|.$$

Deduce that the map $K \longmapsto C_t(T,K;S)$ is continuous and Lebesgue–almost everywhere differentiable.

Assume without loss of generality that $K_1 \geq K_2$**, and let us constitute the following two portfolios**

- P_1**: one Call with strike** K_1 **and** K_1 **zero–coupon bonds with maturity** T**.**
- P_2**: one Call with strike** K_2 **and** K_2 **zero–coupon bonds with maturity** T**.**

The values of P_1 **and** P_2 **at time** t **are respectively**

$$C_t(T,K_1;S) + K_1B(t,T), \text{ and } C_t(T,K_2;S) + K_2B(t,T),$$

while their values at time T **are respectively**

$$(S_T - K_1)^+ + K_1, \text{ and } (S_T - K_2)^+ + K_2.$$

However, we have

$$(S_T - K_2)^+ + K_2 - (S_T - K_1)^+ - K_1 = \begin{cases} 0, \text{ if } S_T \geq K_1, \\ S_T - K_1 < 0, \text{ if } K_2 \leq S_T < K_1, \\ K_2 - K_1 \leq 0, \text{ if } S_T < K_2. \end{cases}$$

Therefore, the value of P_2 **at time** T **is always less than the value of** P_1 **at time** T**, which implies by the no–dominance principle that**

$$C_t(T,K_1;S) + K_1B(t,T) \geq C_t(T,K_2;S) + K_2B(t,T),$$

which is the desired inequality by symmetry of the choice of K_1 and K_2.

We just proved that the map $K \longmapsto C_t(T, K; S)$ was Lipschitz–continuous on $\mathbb{R}_+$ with Lipschitz constant $B(t, T)$. This implies immediately that this map is continuous on $\mathbb{R}_+$. Notice also that this implies that it is differentiable almost–everywhere on $\mathbb{R}_+$.

3) Prove that we have for any $(T, t) \in [0, +\infty) \times [0, T]$

$$\lim_{K \to +\infty} C_t(T, K; S) = 0.$$

Fix some $\varepsilon > 0$ and some $t \in [0, T]$. Since it is clear that $\lim_{K \to +\infty} C_T(T, K; S) = \lim_{K \to +\infty}(S_T - K)^+ = 0$, we know that there exists some $K_0 > 0$ such that for any $K \geq K_0$, we have

$$0 \leq C_T(T, K; S) \leq \frac{\varepsilon}{B(t, T)}. \tag{0.1}$$

Consider now the two following portfolios

- **P_1: one Call with strike K. Value at t is $C_t(T, K; S)$.**
- **P_2: $\varepsilon / B(t, T)$ zero–coupon bonds with maturity T. Value at t is ε.**

Inequality (1) implies that for any $K \geq K_0$, the value of portfolio P_1 at T is always below the value of portfolio P_2 at time T. By the no–dominance principle this implies that for any $K \geq K_0$

$$0 \leq C_t(T, K; S) \leq \varepsilon,$$

which implies the desired result.

2. **Put options properties**

Prove Proposition 1.4.19 from the lecture notes.

(i) **We use Call–Put parity implying**

$$P_t(T, K; S) = C_t(T, K; S) - S_t + KB(t, T).$$

Since we also have

$$\big(S_t - KB(t, T)\big)^+ \leq C_t(T, K; S) \leq S_t,$$

we deduce that $P_t(T, K; S) \leq KB(t, T)$ and

$$P_t(T, K; S) \geq \big(S_t - KB(t, T)\big)^+ - S_t + KB(t, T) \geq KB(t, T) - S_t.$$

Since the above also shows that $P_t(T, K; S) \geq 0$, we deduce that the remaining inequality holds.

(ii) **Convexity is immediate from Call–Put parity and the convexity for European Call options. For monotonicity, fix some $(K_1, K_2) \in \mathbb{R}_+^2$ with $K_1 \geq K_2$. We constitute the following two portfolios**

- **P_1: one Put with strike K_1.**
- **P_2: one Put with strike K_2.**

The values of P_1 and P_2 at time t are respectively $P_t(T, K_1; S)$ and $P_t(T, K_2; S)$, while their values at time T are respectively $(K_1 - S_T)^+$ and $(K_2 - S_T)^+$. The value of P_1 at T is obviously always above that of P_2 at T, since $K_1 \geq K_2$, which allows us to conclude using the no–dominance principle, that the map $K \longmapsto P_t(T, K; S)$ is non–decreasing.

(iii) **Assume without loss of generality that $K_1 \geq K_2$, and let us constitute the following two portfolios**

- P_1: one Put with strike K_1 and K_2 zero–coupon bonds with maturity T.
- P_2: one Put with strike K_2 and K_1 zero–coupon bonds with maturity T.

The values of P_1 and P_2 at time t are respectively

$$P_t(T, K_1; S) + K_2 B(t, T), \text{ and } P_t(T, K_2; S) + K_1 B(t, T),$$

while their values at time T are respectively

$$(K_1 - S_T)^+ + K_2, \text{ and } (K_2 - S_T)^+ + K_1.$$

However, we have

$$(K_1 - S_T)^+ + K_2 - (K_2 - S_T)^+ - K_1 = \begin{cases} 0, \text{ if } S_T < K_2, \\ K_2 - S_T \leq 0, \text{ if } K_2 \leq S_T < K_1, \\ K_2 - K_1 \leq 0, \text{ if } S_T \geq K_1. \end{cases}$$

Therefore, the value of P_1 at time T is always less than the value of P_2 at time T, which implies by the no–dominance principle that

$$P_t(T, K_1; S) + K_2 B(t, T) \geq P_t(T, K_2; S) + K_1 B(t, T),$$

which is the desired inequality by symmetry of the choice of K_1 and K_2.

We just proved that the map $K \longmapsto P_t(T, K; S)$ was Lipschitz–continuous on $\mathbb{R}_+$ with Lipschitz constant $B(t, T)$. This implies immediately that this map is continuous on $\mathbb{R}_+$. Notice also that this implies that it is differentiable almost–everywhere on $\mathbb{R}_+$.

(iv) This is immediate from the inequality $P_t(T, K; S) \geq \big(KB(t, T) - S_t\big)^+$.

(v) Let us start with the left inequality. Assume to the contrary that with positive probability

$$P_t^{\mathrm{A}}(T, K_2; S) - P_t^{\mathrm{A}}(T, K_1; S) < 0,$$

and then implement the strategy consisting in doing nothing for any realisation of the world such that the above inequality does not hold, and in the other cases in buying at time t the American Put with strike K_2 and sell the one with strike K_1. Our wealth is positive so that we use it to buy zero–coupon bonds with maturity T. At the time $\tau \in [t, T]$ when the American Put we sold is exercised, we exercise the other one and sell our bonds. Our wealth is then

$$\frac{P_t^{\mathrm{A}}(T, K_1; S) - P_t^{\mathrm{A}}(T, K_2; S)}{B(t, \tau)} + (K_2 - S_\tau)^+ - (K_1 - S_\tau)^+ > 0,$$

since $K_2 \geq K_1$. This is an arbitrage opportunity, which proves the inequality.

Assume now that with positive probability

$$P_t^{\mathrm{A}}(T, K_2; S) - P_t^{\mathrm{A}}(T, K_1; S) > K_2 - K_1,$$

and implement the strategy consisting in doing nothing for any realisation of the world such that the above inequality does not hold, and in the other cases in selling one American Put with strike K_2, buy one with strike K_1 and buy $(K_2 - K_1)/B(t, T)$ zero–coupon bonds with maturity T. Our wealth is then by assumption positive, and we use it to buy more zero–coupon bonds with maturity T. At the time $\tau \in [t, T]$ when the American Put we sold is exercised, we exercise the other one and sell our bonds. Our wealth is then

$$\frac{P_t^{\mathrm{A}}(T, K_2; S) - P_t^{\mathrm{A}}(T, K_1; S) - K_2 + K_1}{B(t, \tau)} + \frac{K_2 - K_1}{B(t, \tau)} + (K_1 - S_\tau)^+ - (K_2 - S_\tau)^+.$$

The first term is positive, while the sum of the 3 remaining ones is equal to

$$\begin{cases} \dfrac{K_2-K_1}{B(t,\tau)} \geq 0, \text{ if } S_\tau \geq K_2, \\ \dfrac{K_2-K_1}{B(t,\tau)} - K_2 + S_\tau \geq \dfrac{K_2-K_1}{B(t,\tau)}(1-B(t,\tau)) \geq 0, \text{ if } K_1 \leq S_\tau < K_2, \\ \dfrac{K_2-K_1}{B(t,\tau)}(1-B(t,\tau)) \geq 0, \text{ if } S_\tau < K_1. \end{cases}$$

We thus once again have an arbitrage opportunity, which proves the desired result.

(vi) Let us start by proving for any $\varepsilon > 0$ that

$$\frac{P_t^{\mathrm{A}}(T,K+\varepsilon;S) - P_t^{\mathrm{A}}(T,K;S)}{\varepsilon} - \frac{P_t^{\mathrm{A}}(T,K;S)}{K} \geq \frac{P_t(T,K+\varepsilon;S) - P_t(T,K;S)}{\varepsilon} - \frac{P_t(T,K;S)}{K}. \tag{0.2}$$

Let us assume to the contrary that with positive probability

$$\frac{P_t^{\mathrm{A}}(T,K+\varepsilon;S) - P_t^{\mathrm{A}}(T,K;S)}{\varepsilon} - \frac{P_t^{\mathrm{A}}(T,K;S)}{K} < \frac{P_t(T,K+\varepsilon;S) - P_t(T,K;S)}{\varepsilon} - \frac{P_t(T,K;S)}{K},$$

and let us consider the following strategy consisting in doing nothing for any realisation of the world such that the above inequality does not hold, and in the other cases

- At time t, we
  - sell $1/\varepsilon$ European Puts with strike $K+\varepsilon$, and sell $\frac{\varepsilon+K}{\varepsilon K}$ American Puts with strike K,
  - buy $1/\varepsilon$ American Puts with strike $K+\varepsilon$, and buy $\frac{K+\varepsilon}{\varepsilon K}$ European Puts with strike K.

• If at any time $\tau \in [t,T)$ the buyer of our $\frac{K+\varepsilon}{\varepsilon K}$ American Puts with strike K decides to exercise them, we immediately exercise our $1/\varepsilon$ American Puts with strike $K+\varepsilon$, if $S_\tau < K$. If $S_\tau \geq K$, we do nothing. The associated wealth is non–negative, and is equal to $\frac{S_\tau}{K}$. Then, we buy $1/K$ assets. Overall, we are left with 0.

If there is no early exercise, we do not do anything.

- At time T, two cases are possible
  - if there was early exercise at τ, then our final wealth is, when we had $S_\tau < K$

$$\frac{S_T}{K} + \frac{K+\varepsilon}{\varepsilon K}(K-S_T)^+ - \frac{1}{\varepsilon}(K+\varepsilon-S_T)^+ = \begin{cases} \dfrac{S_T}{K} > 0, \text{ if } S_T \geq K+\varepsilon, \\ \dfrac{(K+\varepsilon)(S_T-K)}{\varepsilon K} \geq 0, \text{ if } K \leq S_T < K+\varepsilon, \\ 0, \text{ if } 0 < S_T < K, \end{cases}$$

and when we had $S_\tau \geq K$

$$\frac{K+\varepsilon}{\varepsilon K}(K-S_T)^+ \geq 0.$$

  - If there was no early exercise, then our final wealth is 0.

In any case, our final wealth is always non–negative and is positive in the cases where $S_\tau < K$ and $S_T \geq K+\varepsilon$, or $S_\tau \geq K$ and $S_T < K$. We thus have an arbitrage opportunity, which is absurd and proves the desired inequality. We can then conclude using the continuity and the monotonicity of P^{A} from (v), which allows to let ε got to 0 in (2).