Introduction to Mathematical Finance
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Assignment 4 (Solutions)

1. **Call-Call and Call-Put options**

Consider a financial market without arbitrages. We consider a Call-Call option, that is to say an option whose underlying is itself an option. More precisely, a Call-Call option with strike K_1 and maturity T_1 on a Call option with strike K_2 and maturity $T_2 > T_1$ and underlying S has the following payoff at T_1

$$CC_{T_1}(K_1, T_1; K_2, T_2, S) := \big(C_{T_1}(T_2, K_2; S) - K_1\big)^+.$$

Similarly, we define Put–Call option with strike K_1 and maturity T_1 on a Call option with strike K_2 and maturity $T_2 > T_1$ and underlying S has having the following payoff at T_1

$$PP_{T_1}(K_1, T_1; K_2, T_2, S) := \big(K_1 - C_{T_1}(T_2, K_2; S)\big)^+.$$

Prove the following parity relationship for any $t \in [0, T_1]$

$$CC_t(K_1, T_1; K_2, T_2, S) - PP_t(K_1, T_1; K_2, T_2, S) = C_t(T_2, K_2; S) - K_1 B(t, T_1).$$

The proof is immediate by following the same arguments that we used in class, simply replacing the underlying asset value S_t by the appropriate one here, that is to say $C_t(T_2, K_2; S)$.

2. **Bull Spread**

A Bull Spread consist in buying a call with strike K_1 and selling a call with strike $K_2 > K_1$.

1) Does this strategy have an initial cost?

The cost of the strategy is initially

$$C_0(T, K_1; S) - C_0(T, K_2; S),$$

which is positive since $K_2 > K_1$.

2) Calculate and draw the corresponding payoff.

The payoff of the strategy is at time T

$$(S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} K_2 - K_1, \ \textbf{if } S_T \geq K_2, \\ S_T - K_1, \ \textbf{if } K_1 \geq S_T < K_2, \\ 0, \ \textbf{if } S_T < K_1. \end{cases}$$

3) What is the interest of this strategy?

You should notice that the payoff is similar to that of a Call option with strike K_1. The main difference is that the payoff remains constant when $S_T \geq K_2$. Compared to a Call with strike K_1, the initial cost is clearly less large. The Bull spread strategy is therefore useful for an investor anticipating that the value of S is going to increase, but will stay in the range $[K_1, K_2]$ at the horizon T (while the Call with strike K_1 only anticipates that the value of S will increase).

3. **Bottom Straddle**

A Bottom Straddle consists in buying a call and a put with the same strike and the same maturity. Same questions as in the previous exercise.

The cost of the strategy is initially

$$C_0(T, K; S) + P_0(T, K; S),$$

which is obviously positive.

The payoff of the strategy is at time T

$$(S_T - K)^+ + (K - S_T)^+ = |S_T - K|.$$

Such a strategy makes benefits as soon as the value of S_T is far away from K. It is therefore useful for an investor anticipating a very volatile market, where S will have large variations, potentially in both directions.

4. **Butterfly Spread**

A Butterfly Spread consists in selling 2 calls with strike K_2 and buying a Call with strike K_1 and a call with strike K_3, with $0 \leq K_2 - K_1 = K_3 - K_2$. Same questions as in the previous exercise.

The cost of the strategy is initially

$$C_0(T, K_1; S) + C_0(T, K_3; S) - 2C_0(T, K_2; S).$$

Notice that $K_2 = (K_1 + K_3)/2$, so that by convexity of Call prices with respect to strike, we deduce that the initial cost is positive.

The payoff of the strategy is at time T

$$(S_T - K_1)^+ + (S_T - K_3)^+ - 2(S_T - K_2)^+ = \begin{cases} 0, \text{ if } S_T \geq K_3, \\ K_3 - S_T, \text{ if } K_2 \geq S_T < K_3, \\ S_T - K_1, \text{ if } K_1 \geq S_T < K_2, 0, \text{ if } S_T < K_1. \end{cases}$$

Such a strategy makes benefits as soon as the value of S_T stays close to K_2. It is therefore useful for an investor anticipating that S will have very small variations around K_2.

5. **Barrier option**

We consider a Up and Out Call i.e. an option with payoff

$$(S_T - K)^+ \mathbf{1}_{\{\sup_{0 \leq t \leq T} S_t \leq L\}}.$$

1) Explain the name 'barrier option'.

   The name stems from the fact that the option is deactivated if the value of the underlying reaches the upper barrier L between 0 and T. In the nomenclature given in class, this is an Up–and–Out Call option.

2) What happens when $L \leq K$ or $S_0 > L$? We assume now that $L > K$ and $S_0 < L$.

   In both cases the payoff of the option is always 0, so that by the no–dominance principle its value at any prior time will also always be 0.

3) Show that if the price of this option is above $\frac{L-K}{L} S_0$, there exists an arbitrage opportunity.

Assume thus that the value V_0 of the option satisfies

$$V_0 > \frac{L-K}{L} S_0.$$

Let us then use the following strategy

- **At time 0, we sell the option, buy $(L-K)/L$ assets, and use the remaining positive amount to buy Zero–Coupon bonds with maturity T.**
- **At time T, our wealth is**

$$\frac{L-K}{L} S_T - (S_T - K)^+ 1_{\{\sup_{0 \le t \le T} S_t \le L\}} + \frac{V_0 - \frac{L-K}{L} S_0}{B(0,T)}.$$

  Notice then that when $S_T \ge L$

$$\frac{L-K}{L} S_T - (S_T - K)^+ \mathbf{1}_{\{\sup_{0 \le t \le T} S_t \le L\}} = \frac{L-K}{L} S_T \ge 0.$$

  Furthermore when $S_T < L$,

$$\begin{aligned} \frac{L-K}{L} S_T - (S_T - K)^+ \mathbf{1}_{\{\sup_{0 \le t \le T} S_t \le L\}} &\ge \frac{L-K}{L} S_T - (S_T - K)^+ \\ &= \begin{cases} \frac{K}{L}(L - S_T) \ge 0, \ \textbf{if } K \le S_T < L, \\ \frac{L-K}{L} S_T \ge 0, \ \textbf{if } S_T < K. \end{cases} \end{aligned}.$$

  Since

$$\frac{V_0 - \frac{L-K}{L} S_0}{B(0,T)} > 0,$$

  we deduce that this strategy is an arbitrage opportunity.