Introduction to Mathematical Finance
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Assignment 5 (solutions)

1. **On American Calls with dividends**

Let S be a stock paying a known dividend amount κ at some known time $\tau \in (0,T)$.

1) Prove that the following relationship holds for any $t < \tau$

$$S_t \geq C_t^A(T,K;S) \geq \max\Big\{\big(S_t - KB(t,\tau)\big)^+, \big(S_t - \kappa B(t,\tau) - KB(t,T)\big)^+\Big\}.$$

The left inequality is easy. Indeed, a portfolio containing one asset, even without dividends, is always more valuable than one containing the American Call, since exercising the American Call always provides less than selling an asset.

The second inequality is a bit more involved. First of all, since we already know that the value of the American Call is non–negative (it is clear since the payoff of this option is non–negative), it is enough to prove

$$C_t^A(T,K;S) \geq S_t - KB(t,\tau), \text{ and } C_t^A(T,K;S) \geq S_t - \kappa B(t,\tau) - KB(t,T).$$

Let us start with the first inequality. Assume to the contrary that

$$C_t^A(T,K;S) < S_t - KB(t,\tau),$$

and then buy at time t the American Call and K Zero–Coupon bonds with maturity τ, and sell one asset. The net wealth is positive by assumption and we invest it in Zero–Coupon bonds.

At time τ, just before the dividend payment is made, we buy back the asset, exercise the American Call and sell our Zero–Coupon bonds. Our net wealth at τ is thus

$$\begin{aligned}&(S_\tau - K)^+ - S_\tau + K + \frac{S_t - Kd(t,\tau) - C_t^A(T,K;S)}{B(t,\tau)}\\ &= (K - S_\tau)^+ + \frac{S_t - KB(t,\tau) - C_t^A(T,K;S)}{B(t,\tau)} > 0.\end{aligned}$$

Hence we have an arbitrage opportunity.

Let us assume now that

$$C_t^A(T,K;S) < S_t - \kappa B(t,\tau) - KB(t,T),$$

and let us buy at time t the American Call, κ Zero–Coupon bonds with maturity τ and K Zero–Coupon bonds with maturity T, and let us sell one asset. The remaining wealth is positive by assumption and we use it to buy Zero–Coupon bonds with maturity T. At time τ we receive κ from our Zero–Coupon bonds, which we use to pay the dividends for the asset we sold. Then at time T, we exercise the American Call and buy back the asset. Our net wealth is

$$(S_T - K)^+ - S_T + K + \frac{S_t - \kappa B(t,\tau) - KB(t,T) - C_t^A(T,K;S)}{B(t,T)} > 0,$$

and we again have an arbitrage opportunity.

2) Assume now that S pays n known dividends $(\kappa_i)_{1\leq i\leq n}$ at the known dates $(\tau_i)_{1\leq i\leq n}$ with $0 < \tau_1 < \tau_2 < \cdots < \tau_n < T$. Prove that for any $t < \tau_1$

$$S_t \geq C_t^A(T,K;S) \geq \max_{0\leq i\leq n}\left\{\left(S_t - KB(t,\tau_{i+1}) - \sum_{j=1}^{i}\kappa_j B(t,\tau_j)\right)^+\right\},$$

with the convention that $\tau_{n+1} = T$ and that a sum over an empty set is 0.

The left inequality is the same as before. For the right one, it is again enough to prove that for all $i = 0, \dots, n$, we have

$$C_t^A(T, K; S) \geq S_t - KB(t, \tau_{i+1}) - \sum_{j=1}^{i} \kappa_j B(t, \tau_j).$$

Assume to the contrary that

$$C_t^A(T, K; S) < S_t - KB(t, \tau_{i+1}) - \sum_{j=1}^{i} \kappa_j B(t, \tau_j),$$

and buy at time t the American Call, K Zero–Coupon bonds with maturity τ_{i+1} and for every $j = 1, \dots, i$, κ_j Zero–Coupon bonds with maturity τ_j (the latter of course do not exist when $i = 0$), and sell one asset. The remaining wealth is positive by assumption and we use it to buy Zero–Coupon bonds with maturity τ_{i+1}. Then, for any $j = 1, \dots, i$, we receive at time τ_j the quantity κ_j which we use to pay the dividends from the asset we sold at time t. We wait until time τ_{i+1}, at which we exercise the American Call and buy back the asset right before the dividend payment. At τ_{i+1}, our wealth is thus

$$K + (S_T - K) - S_T + \frac{S_t - KB(t, \tau_{i+1}) - \sum_{j=1}^{i} \kappa_j B(t, \tau_j) - C_t^A(T, K; S)}{B(t, \tau_{i+1})} > 0,$$

hence we have an arbitrage opportunity.

3) Prove that the only dates where it can be optimal to exercise a Call option on an underlying asset paying n known dividends $(\kappa_i)_{1 \leq i \leq n}$ at the known dates $(\tau_i)_{1 \leq i \leq n}$ are the maturity T, or at the times $(\tau_i)_{1 \leq i \leq n}$, just before the dividends are paid. Under which condition(s) is it not optimal to exercise the American Call prior to T?

First of all, it is clearly never optimal to exercise the Call at any time $t_o \in (\tau_n, T)$, since after the last dividend payment τ_n is made, the Call becomes a Call on a non–dividend paying underlying asset, and we have seen in class that these should never be exercised early.

Now exercising at time $t_o \in (\tau_{j-1}, \tau_j]$ in–between two dividend payments dates can only be optimal at time τ_j, just before the payment of the dividend. Indeed, adapting the proof of the previous question, we have the inequality

$$C_{t_o}^A(T, K; S) \geq \max_{0 \leq i \leq n+1-j} \left\{ \left(S_{t_o} - KB(t_o, \tau_{i+j}) + \sum_{k=1}^{i} \kappa_{k-1+j} B(t_o, \tau_{k-1+j}) \right)^+ \right\}.$$

In particular, this implies that for $i = 0$

$$C_{t_o}^A(T, K; S) \geq \left(S_{t_o} - KB(t_o, \tau_j) \right)^+ \geq \left(S_{t_o} - K \right)^+,$$

where the right–hand side corresponds to exercising the American Call at time t_o. Furthermore, it is easy to check that equality above can only happen if $t_o = \tau_j$, just before the dividend payment, because interest rates are positive. Therefore, the only times where it could make sense to exercise the American option are the maturity T, or the times $(\tau_i)_{1 \leq i \leq n}$, just before the dividends are paid.

Let us now examine under which conditions early exercise could actually be optimal or not. The decision to exercise involves a trade–off between dividend income and interest income. To illustrate this more precisely, consider a stock that pays a single dividend prior to expiration. Just prior to the dividend date the value of an in–the–money American Call, if exercised, is its

intrinsic value $S_{\tau_1} - K$. Just after the dividend is paid, the value of the stock drops to $S_{\tau_1} - \kappa_1$, and since there are no more dividends to be paid after τ_1, we know that, just after the dividend payment, we must have the lower bound

$$C^A_{\tau_1}(T, K; S) \geq \big(S_{\tau_1} - \kappa_1 - KB(\tau_1, T)\big)^+.$$

Hence, a sufficient condition for early exercise to not be optimal at time τ_1 is

$$S_{\tau_1} - K \leq S_{\tau_1} - \kappa_1 - KB(\tau_1, T) \Longleftrightarrow \kappa_1 \leq K\big(1 - B(\tau_1, T)\big).$$

The above equation states that early exercise will not be optimal if the dividend paid at time τ_1 is less than the interests generated by the strike K over the time–period $[\tau_1, T]$.

The above reasoning can be iterated in the case of several dividend payments, and we end up with the property that early exercise for the American Call is not optimal if, at each dividend payment dates, the present value of all the dividend payments remaining (including the present one) is less than the present value of interests earned on the strike until time T. In other words, we must have for any $j = 1, \ldots, n$

$$\sum_{k=j}^{n} \kappa_k B(\tau_j, \tau_k) \leq K\big(1 - B(\tau_j, T)\big).$$

2. **Asymptotics for the CRR model**

We consider a variation on the multi–period binomial model from Section 2.3.2.2. We let our time horizon be some time $T > 0$, and take $\Omega := \{\omega^u, \omega^d\}^m$, where m is a positive integer representing the number of periods in the market. As usual, we fix $\mathcal{F} := \mathcal{P}(\Omega)$. We depart a little bit from the lecture notes' notations, and consider that the market trading dates are given by $(t^m_k)_{k\in\{0,\ldots,m\}}$, where

$$t^m_k := \frac{kT}{m},\ k \in \{0, \ldots, m\}.$$

The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is again given by

$$\mathbb{P}[\{\omega\}] = p^{U(\omega)}(1-p)^{m-U(\omega)},\ \forall \omega := (\omega_1, \ldots, \omega_m) \in \Omega,$$

where $p \in (0, 1)$, and where for any $\omega \in \Omega$, $U(\omega)$ counts the number of elements of ω which are equal to ω^u. We define the filtration $\mathbb{F} := (\mathcal{F}_k)_{k\in\{0,\ldots,m\}}$ by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_m := \mathcal{F}$ and

$$\mathcal{F}_k := \sigma\big((\omega_1, \ldots, \omega_s) : s \in \{1, \ldots, k\}\big),\ k \in \{1, \ldots, m-1\}.$$

The non–risky asset values are given by

$$S^{m,0}_{t^m_k}(\omega) := \left(1 + \frac{rT}{m}\right)^k,\ k \in \{0, \ldots, m\},\ \omega \in \Omega,$$

where $r \in \mathbb{R}$, while that of the risky asset are given, for any $\omega \in \Omega$, by

$$S^m_{t^m_0}(\omega) := S_0,$$

$$S^m_{t^m_{k+1}}(\omega) = S^m_{t^m_{k+1}}(\omega_1, \ldots, \omega_{k+1}) := \begin{cases}(1 + h_m)S^m_{t^m_k}(\omega_1, \ldots, \omega_k),\ \text{if } \omega_{k+1} = \omega^u,\\ (1 + \ell_m)S^m_{t^m_k}(\omega_1, \ldots, \omega_k),\ \text{if } \omega_{k+1} = \omega^d,\end{cases}\quad k \in \{0, \ldots, m-1\},$$

where

$$\ell_m := \frac{rT}{m} - \sigma_-\sqrt{\frac{T}{m}},\ h_m := \frac{rT}{m} + \sigma_+\sqrt{\frac{T}{m}},$$

for some given $(\sigma_-,\sigma_+)\in(0,+\infty)^2$. Notice that we index the asset values by $m\in\mathbb{N}\setminus\{0\}$ since the aim of the exercise is to let m go to $+\infty$.

We will assume throughout that m is large enough so that

$$\ell_m>-1,\ \frac{rT}{m}>-1.$$

1) Prove that for m large enough this market admits no–arbitrage opportunities, and that there is a unique risk–neutral measure $\mathbb{Q}^m$. We let $\underline{m}$ be the lowest value of m such that this holds.

   We are here in a multi–period binomial model, for which we know that (NA) is equivalent to having, with the specification chosen here

$$1+\ell_m<1+\frac{rT}{m}<1+h_m,$$

   which is obviously true. The only restriction here is that the returns of the non–risky asset remain positive, that is $rT/m>-1$, and that

$$0<1+\ell_m<1+b_m,$$

   which is equivalent to having $\ell_m>-1$. It is not too hard to check that whenever $\sigma_-<2\sqrt{r}$ (and $r\geq 0$ obviously) this holds for any positive integer m, while when $r<0$ or $\sigma_-\geq 2\sqrt{r}$, this is equivalent to

$$m>\frac{T}{4}\Big(\sigma_-+\sqrt{\sigma_-^2-4r}\Big)^2.$$

   We also know that under these conditions the market is complete, and there is therefore a unique risk–neutral measure $\mathbb{Q}^m$.

2) We define

$$p_m:=\mathbb{Q}^m\left[\left\{\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}=1+h_m\right\}\right],\ k\in\{0,\dots,m-1\}.$$

   Give the exact value of p_m and justify that it indeed does not depend on $k\in\{0,\dots,m-1\}$.

   We know that $\mathbb{Q}^m$ makes the process $\left(\frac{S^m_{t^m_k}}{S^{m,0}_{t^m_k}}\right)_{k\in\{0,\dots,m\}}$ into a martingale. In particular we must have

$$S_0=\mathbb{E}^{\mathbb{Q}^m}\left[\frac{S^m_{t^m_1}}{S^{0,m}_{t^m_1}}\right]=\frac{p_m(1+h_m)S_0+(1-p_m)(1+\ell_m)S_0}{1+\frac{rT}{m}},$$

   from which we deduce that

$$p_m=\frac{\frac{rT}{m}-\ell_m}{h_m-\ell_m}=\frac{\sigma_-}{\sigma_++\sigma_-}.$$

   It is straightforward to check that defining $\mathbb{Q}^m$ by

$$\mathbb{Q}^m[\{\omega\}]=p_m^{U(\omega)}(1-p_m)^{m-U(\omega)},\ \forall\omega:=(\omega_1,\dots,\omega_m)\in\Omega,$$

   does indeed lead us to a risk–neutral measure, exactly as in the lecture notes.

3) Which property does the sequence of random variables $\left(\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}\right)_{k\in\{0,\dots,m-1\}}$ satisfy under $\mathbb{Q}^m$.

   This is an i.i.d. sequence under $\mathbb{Q}^m$.

4) We denote by Φ_m the characteristic function under $\mathbb{Q}^m$ of the random variable $\log\big(S^m_{t_1^m}/S^m_{t_0^m}\big)$, that is to say

$$\Phi_m(\lambda) := \mathbb{E}^{\mathbb{Q}^m}\left[\exp\left(i\lambda\log\left(\frac{S^m_{t_1^m}}{S^m_{t_0^m}}\right)\right)\right].$$

Show that the following Taylor expansion holds

$$\Phi_m(\lambda) = 1 + \left(i\lambda\left(r - \frac{\sigma_-\sigma_+}{2}\right) - \lambda^2\frac{\sigma_-\sigma_+}{2}\right)\frac{T}{m} + \circ\left(\frac{1}{m}\right).$$

We have

$$\begin{aligned}
\Phi_m(\lambda) &= p_m\exp\big(i\lambda\log(1+h_m)\big) + (1-p_m)\exp\big(i\lambda\log(1+\ell_m)\big)\\
&= \frac{\sigma_-}{\sigma_+ + \sigma_-}\exp\left(i\lambda\log\left(1+\sigma_+\sqrt{\frac{T}{m}}+\frac{rT}{m}\right)\right) + \frac{\sigma_+}{\sigma_+ + \sigma_-}\exp\left(i\lambda\log\left(1-\sigma_-\sqrt{\frac{T}{m}}+\frac{rT}{m}\right)\right)\\
&= \frac{\sigma_-}{\sigma_+ + \sigma_-}\exp\left(i\lambda\left(\sigma_+\sqrt{\frac{T}{m}} + \left(r-\frac{\sigma_+^2}{2}\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right)\right)\right)\\
&\quad + \frac{\sigma_+}{\sigma_+ + \sigma_-}\exp\left(i\lambda\left(-\sigma_-\sqrt{\frac{T}{m}} + \left(r-\frac{\sigma_-^2}{2}\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right)\right)\right)\\
&= \frac{\sigma_-}{\sigma_+ + \sigma_-}\left(1 + i\lambda\sigma_+\sqrt{\frac{T}{m}} + \left(i\lambda\left(r-\frac{\sigma_+^2}{2}\right) - \frac{\lambda^2\sigma_+^2}{2}\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right)\right)\\
&\quad + \frac{\sigma_+}{\sigma_+ + \sigma_-}\left(1 - i\lambda\sigma_-\sqrt{\frac{T}{m}} + \left(i\lambda\left(r-\frac{\sigma_-^2}{2}\right) - \frac{\lambda^2\sigma_-^2}{2}\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right)\right)\\
&= 1 + \left(i\lambda\left(r-\frac{\sigma_-\sigma_+}{2}\right) - \frac{\sigma_-\sigma_+}{2}\lambda^2\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right).
\end{aligned}$$

5) Let $Y_m := \log(S^m_T/S_0)$. Express the characteristic function of Y_m under $\mathbb{Q}^m$ in terms of Φ_m, and then show that the sequence $(Y_m)_{m\geq\underline{m}}$ converges in law, under $\mathbb{Q}^m$, to a random variable whose distribution you'll give explicitly.

Hint: It would be useful here to prove the following lemma, which provides a result that you must know for real–valued sequences, but not necessarily for complex–valued ones.

Lemma 0.1. *For any $(z,R)\in\mathbb{C}\times(0,+\infty)$, if $|z|\leq R$, then for any positive integer n*

$$\left|\mathrm{e}^z - \left(1+\frac{z}{n}\right)^n\right| \leq \mathrm{e}^R - \left(1+\frac{R}{n}\right)^n.$$

In addition, for any complex–valued sequence $(z_n)_{n\in\mathbb{N}\setminus\{0\}}$ converging to some $z\in\mathbb{C}$, we have

$$\lim_{n\to+\infty}\left(1+\frac{z_n}{n}\right)^n = \mathrm{e}^z.$$

Notice that

$$Y_m = \log\left(\prod_{k=0}^{m-1}\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}\right) = \sum_{k=0}^{m-1}\log\left(\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}\right).$$

Using the fact that the sequence $\left(\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}\right)_{k\in\{0,\dots,m-1\}}$ **is i.i.d. under $\mathbb{Q}^m$, we deduce that for any** $\lambda\in\mathbb{R}$

$$\mathbb{E}^{\mathbb{Q}^m}\left[\mathrm{e}^{i\lambda Y_m}\right] = \mathbb{E}^{\mathbb{Q}^m}\left[\prod_{k=0}^{m-1}\exp\left(i\lambda\log\left(\frac{S^m_{t^m_{k+1}}}{S^m_{t^m_k}}\right)\right)\right] = \Phi^m_m(\lambda).$$

By the previous question, we know that we can write

$$\Phi_m^m(\lambda) = \left(1 + \frac{z_m}{m}\right)^m,$$

where

$$z_m := \left(i\lambda\left(r - \frac{\sigma_-\sigma_+}{2}\right) - \frac{\sigma_-\sigma_+}{2}\lambda^2\right)T + \varepsilon_m,$$

where $(\varepsilon_m)_{m\geq \underline{m}}$ **is a sequence converging to** 0**. Using the result of the lemma, we thus deduce that**

$$\lim_{m\to+\infty} \mathbb{E}^{\mathbb{Q}^m}\left[\mathrm{e}^{i\lambda Y_m}\right] = \exp\left(i\lambda\left(r - \frac{\sigma_-\sigma_+}{2}\right)T - \frac{\sigma_-\sigma_+}{2}\lambda^2 T\right).$$

This shows that the sequence $(Y_m)_{m\geq \underline{m}}$ **converges in distribution to a Gaussian random variable** N **with mean** μ **and variance** Σ **where**

$$\mu := \left(r - \frac{\sigma_-\sigma_+}{2}\right)T,\ \Sigma := \sigma_-\sigma_+ T.$$

We finish with the

Proof of Lemma 0.1. Using the development of the exponential as an entire series, we have first

$$\left|\mathrm{e}^z - \left(1 + \frac{z}{n}\right)^n\right| = \left|\sum k = 0^{+\infty} \frac{z^k}{k!} - \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{n^k}\right| \leq \sum_{k=0}^{n} |z|^k \left|\frac{1}{k!} - \binom{n}{k}\frac{1}{n^k}\right| + \sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!}$$
$$\leq \sum_{k=0}^{n} R^k \left|\frac{1}{k!} - \binom{n}{k}\frac{1}{n^k}\right| + \sum_{k=n+1}^{+\infty} \frac{R^k}{k!}.$$

Now notice that for any $k \in \{0, \dots, n\}$

$$\binom{n}{k}\frac{1}{n^k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{n^k}\frac{1}{k!} \leq \frac{1}{k!}.$$

Therefore, we have

$$\left|\mathrm{e}^z - \left(1 + \frac{z}{n}\right)^n\right| \leq \sum_{k=0}^{n} R^k \left(\frac{1}{k!} - \binom{n}{k}\frac{1}{n^k}\right) + \sum_{k=n+1}^{+\infty} \frac{R^k}{k!} = \mathrm{e}^R - \left(1 + \frac{R}{n}\right)^n.$$

Now since the sequence converges to z, it must be bounded by some $R > 0$. Using the first part of the lemma, we thus have for any $n \in \mathbb{N} \setminus \{0\}$

$$\left|\mathrm{e}^z - \left(1 + \frac{z_n}{n}\right)^n\right| \leq \left|\mathrm{e}^{z_n} - \left(1 + \frac{z_n}{n}\right)^n\right| + |\mathrm{e}^z - \mathrm{e}^{z_n}| \leq \mathrm{e}^R - \left(1 + \frac{R}{n}\right)^n + |\mathrm{e}^z - \mathrm{e}^{z_n}|,$$

which converges to 0 as n goes to ∞. ☐

6) Prove that for any $K \geq 0$

$$\lim_{m\to+\infty} P_0(T, K; S^m) = \mathrm{e}^{-rT} \int_{\mathbb{R}} \left(K - S_0 \mathrm{e}^{a+bx}\right)^+ \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \mathrm{d}x,$$

where you will explicit the constants a and b in terms of r, σ_-, σ_+, and T.

We have that

$$P_0(T, K; S^m) = \frac{1}{S_T^{0,m}} \mathbb{E}^{\mathbb{Q}^m}\left[(K - S_T^m)^+\right] = \left(1 + \frac{rT}{m}\right)^{-m} \mathbb{E}^{\mathbb{Q}^m}\left[\left(K - S_0 \mathrm{e}^{Y_m}\right)^+\right].$$

Now given the convergence in law we proved in the previous question, and since the map $x \longmapsto (K - S_0\mathrm{e}^x)^+$ is bounded and continuous on $\mathbb{R}$, we have by weak convergence that

$$\lim_{m\to+\infty} P_0(T,K;S^m) = \mathrm{e}^{-rT}\int_{\mathbb{R}} \left(K - S_0\mathrm{e}^{a+bx}\right)^+ \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x,$$

where $a = \mu$ and $b = \sqrt{\Sigma}$.

7) Deduce that there exist constants d_0 and d_1 (which you will again provide explicitly) such that

$$\lim_{m\to+\infty} P_0(T,K;S^m) = \mathrm{e}^{-rT}K\mathcal{N}(-d_0) - S_0\mathcal{N}(-d_1),$$

where $\mathcal{N}(x) := \int_{-\infty}^{x} \frac{\mathrm{e}^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\mathrm{d}u$, $x \in \mathbb{R}$ is the repartition function of a standard Gaussian random variable.

Prove that a similar formula holds for $\lim_{m\to+\infty} C_0(T,K;S^m)$.

Direct computations show that

$$d_1 = \frac{1}{\sqrt{\sigma_-\sigma_+T}}\log\left(\frac{S_0}{\mathrm{e}^{-rT}K}\right) + \frac{1}{2}\sqrt{\sigma_-\sigma_+T},\ d_0 = \frac{1}{\sqrt{\sigma_-\sigma_+T}}\log\left(\frac{S_0}{\mathrm{e}^{-rT}K}\right) - \frac{1}{2}\sqrt{\sigma_-\sigma_+T}.$$

Besides, using the Call–Put parity formula and the symmetry of the Gaussian distribution, we deduce

$$\lim_{m\to+\infty} C_0(T,K;S^m) = S_0\mathcal{N}(d_1) - \mathrm{e}^{-rT}K\mathcal{N}(d_0).$$

8) (Optional question) Redo the whole exercise until question 5)

$$\ell_m := \frac{rT}{m} - \sigma_-\frac{T}{m},\ h_m := \frac{rT}{m} + \sigma_+ - \sigma_-\frac{T}{m}.$$

(Notice that obviously the expansion in 4) will now be different.)

The first change is that now, in order for asset prices to remain positive, we must have $rT/m > -1$ and $m > T(\sigma_- - r)$. Then, the risk–neutral probability is now given by

$$p_m = \frac{\frac{rT}{m} - \ell_m}{h_m - \ell_m} = \frac{\sigma_-}{\sigma_+}\frac{T}{m}.$$

Then, similar computations as before show that

$$\Phi_m(\lambda) = 1 + \left(i\lambda(r - \sigma_-) + \frac{\sigma_-}{\sigma_+}\left(\mathrm{e}^{i\lambda\log(1+\sigma_+)} - 1\right)\right)\frac{T}{m} + \circ\left(\frac{T}{m}\right).$$

Then, the characteristic function under $\mathbb{Q}^m$ of $\log\left(S_T^m/S_0\right)$ is such that

$$\lim_{m\to+\infty} \Phi_m^m(\lambda) = \mathrm{e}^{i\lambda(r-\sigma_-)T}\exp\left(\frac{\sigma_-}{\sigma_+}\left(\mathrm{e}^{i\lambda\log(1+\sigma_+)} - 1\right)T\right),$$

which is the characteristic function of a random variable X of the form

$$X := (r - \sigma_-)T + \log(1+\sigma_+)Y,$$

where Y has a Poisson distribution with parameter $\frac{\sigma_-}{\sigma_+}T$.