Introduction to Mathematical Finance
Dylan Possamaï

Assignment 7 (solutions)

About hedging

We consider a T-period binomial market with a risk-less asset with constant return $R > 0$. This means in particular that

$$S_t^0 = R^t,\ t \in \{0, \dots, T\}.$$

There is only one risky asset. At time 0, it is worth $S_0 \in (0, +\infty)$ and there is $0 < d < u$ such that

$$S_{t+1}(\omega) = \big(\mathbf{1}_{\{\omega=\omega^u\}} u + \mathbf{1}_{\{\omega=\omega^d\}} d\big) S_t,\ t \in \{0, \dots, T-1\}.$$

1)a) Recall the condition ensuring that (**NA**) holds in this market.

We know that for any binomial model, the no-arbitrage condition is equivalent to the return of the no-risky asset being strictly between the lowest and the highest values of the return of the risky asset. Here this reads

$$d < R < u.$$

In this case, there is a unique risk-neutral measure $\mathbb{Q}$ (so that the market is also complete), given by

$$\mathbb{Q}\big[\{\omega\}\big] = q^{U(\omega)}(1-q)^{T-U(\omega)},\ \omega \in \Omega,$$

where for any $\omega \in \Omega$, $U(\omega)$ gives the number of components of ω equal to ω^u, and where

$$q := \frac{R-d}{u-d}.$$

1)b) Let us be given a European option with maturity T, with payoff $h(S_T)$ for some map $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, and we denote by p_t the price process for this option (that is $p_t(\omega)$ is the value of this option at time t when the realisation of the world is $\omega \in \Omega$. Prove that it is possible to find a map $v : \{0, \dots, T\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$p_t(\omega) = v\big(t, S_t(\omega)\big),\ (t, \omega) \in \{0, \dots, T\} \times \Omega.$$

In particular, you will give a recursive procedure allowing to compute v.

We have seen during the lectures that we could replicate options in the multiperiod binomial model by defining recursively backward, for any $\omega := (\omega_1, \dots, \omega_T) \in \Omega$, the following $\mathbb{F}$-adapted process $V := \{V_t : t \in \{0, \dots, T\}\}$

$$V_T(\omega_1, \dots, \omega_T) := h\big(S_T(\omega_1, \dots, \omega_T)\big),\ V_t(\omega_1, \dots, \omega_t) := \mathbb{E}^{\mathbb{Q}}\left[\frac{V_{t+1}}{R}\bigg|\mathcal{F}_t\right](\omega_1, \dots, \omega_t),\ t \in \{0, \dots, T-1\}.$$

Then if we define the strategy $\Delta \in \mathcal{A}(\mathbb{R})$, for any $\omega := (\omega_1, \dots, \omega_T) \in \Omega$, by

$$\Delta_t(\omega_1, \dots, \omega_t) := \frac{V_{t+1}(\omega_1, \dots, \omega_t, \omega^u) - V_{t+1}(\omega_1, \dots, \omega_t, \omega^d)}{(u-d)S_t(\omega_1, \dots, \omega_t)},\ t \in \{0, \dots, T-1\},$$

we have for any $t \in \{0, \dots, T\}$

$$\mathbb{P}\big[X_t^{V_0, \Delta} = V_t\big] = 1,$$

and in particular, (V_0, Δ) is a replicating strategy for the payoff ξ.

As such, the value of the option at any time $t \in \{0, \dots, T\}$ is given by V_t. In the specific case of an European option, there are further simplifications, and we will now show via backward induction that

$$V_t(\omega) = v\big(t, S_t(\omega)\big),\ (t, \omega) \in \{0, \dots, T\} \times \Omega,$$

for some map v. For $t = T$, we have

$$V_T(\omega) = h\big(S_T(\omega)\big),\ \omega \in \Omega,$$

so that the result is true. Assuming then that the result is true for some $t+1 \in \{1, \dots, T\}$, we have

$$V_t(\omega) := \mathbb{E}^{\mathbb{Q}}\left[\frac{V_{t+1}}{R}\middle|\mathcal{F}_t\right](\omega) = \mathbb{E}^{\mathbb{Q}}\left[\frac{v(t+1, S_{t+1})}{R}\middle|\mathcal{F}_t\right](\omega) = \frac{1}{R}\Big(qv\big(t+1, uS_t(\omega)\big) + (1-q)v\big(t+1, dS_t(\omega)\big)\Big).$$

This proves the desired result, and the map v is defined recursively by

$$v(T, x) := h(x),\ v(t, x) = \frac{1}{R}\big(qv(t+1, ux) + (1-q)v(t+1, dx)\big),\ (t, x) \in \{0, \dots, T-1\} \times \mathbb{R}_+.$$

$1)c)$ Let $(x, \Delta) \in \mathbb{R}_+ \times \mathcal{A}(\mathbb{R})$ be a replication strategy for the aforementioned European option. Show that you can find a map $\varphi : \{0, \dots, T-1\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+^\star$ such that

$$\Delta_t(\omega) = \varphi\big(t, S_t(\omega)\big),\ (t, \omega) \in \{0, \dots, T-1\} \times \Omega.$$

Given what we recalled above, we have

$$\Delta_t(\omega_1, \dots, \omega_t) := \frac{V_{t+1}(\omega_1, \dots, \omega_t, \omega^u) - V_{t+1}(\omega_1, \dots, \omega_t, \omega^d)}{(u-d)S_t(\omega_1, \dots, \omega_t)},\ t \in \{0, \dots, T-1\}.$$

Therefore, by the previous question

$$\Delta_t(\omega) = \frac{v(t+1, uS_t(\omega)) - v(t+1, dS_t(\omega))}{(u-d)S_t(\omega)},$$

so that the map φ exists indeed and is given by

$$\varphi(t, x) := \frac{v(t+1, ux) - v(t+1, dx)}{(u-d)x},\ (t, x) \in \{0, \dots, T-1\} \times \mathbb{R}_+.$$

$1)d)$ Show that if the map $x \longmapsto h(x)$ is monotone, then for any $t \in \{0, \dots, T\}$, the map $x \longmapsto v(t, x)$ has the same monotony. Deduce that that whenever $x \longmapsto h(x)$ is non-decreasing, $\varphi \geq 0$, and whenever $x \longmapsto h(x)$ is non-increasing, then $\varphi \leq 0$. How can you interpret this result?

Again, this is simply a question of arguing by induction. For $t = T$, we have $v(T, \cdot) = h(\cdot)$, so that the result is obvious. Assume now that for some $t+1 \in \{1, \dots, T\}$ the result holds. We have

$$v(t, \cdot) = \frac{1}{R}\big(qv(t+1, u\cdot) + (1-q)v(t+1, d\cdot)\big),$$

from which the result is immediate since $q \in (0,1)$ and u, d, and R are positive. As for the sign of φ, we have

$$\varphi(t, x) := \frac{v(t+1, ux) - v(t+1, dx)}{(u-d)x}.$$

Hence, since $u > d$ and $x \in \mathbb{R}_+$, when v is non–increasing, $\varphi \leq 0$, and when v is non–decreasing $\varphi \geq 0$.

The result we obtained means that for replicating European options with non–decreasing (resp. non–increasing) payoffs, one should always be long (resp. short) the risky asset. This is an intuitive result in the sense that when one has sold such an option with a non–decreasing (resp. non–increasing) payoff, an increase of the value of the asset is a bad (resp. good) outcome. The replicating strategy, which is then a hedging strategy too, should make sure to compensate losses associated to the option when the risky asset price increases (resp. decreases), which means buying (resp. selling) the risky asset.

2) We suppose throughout this question that $x \longmapsto h(x)$ is convex.

2)a) Show that for any $t \in \{0, \dots, T\}$, the map $x \longmapsto v(t,x)$ is also convex.

Again, this a simple backward induction. For $t = T$, we have $v(T,\cdot) = h(\cdot)$, so that the result is obvious. Assume now that for some $t+1 \in \{1, \dots, T\}$ the result holds. We have

$$v(t,\cdot) = \frac{1}{R}\big(qv(t+1,u\cdot) + (1-q)v(t+1,d\cdot)\big),$$

from which the result is immediate since $q \in (0,1)$ and u, d, and R are positive.

2)b) Show that for any $(x,y,z) \in \mathbb{R}_+^3$ such that $x < y < z$, we have

$$\frac{h(y)-h(x)}{y-x} \leq \frac{h(z)-h(x)}{z-x} \leq \frac{h(z)-h(y)}{z-y}.$$

Define $\lambda := \frac{y-x}{z-x}$ and notice that $\lambda \in (0,1)$ since $0 < x < y < z$. Using the convexity of h, we have

$$h\big(\lambda z + (1-\lambda)x\big) \leq \lambda h(z) + (1-\lambda)h(x).$$

Now since $\lambda z + (1-\lambda)x = y$, we have proved that

$$h(y) \leq \frac{y-x}{z-x}h(z) + \frac{z-y}{z-x}h(x), \text{ which is equivalent to } \frac{y-x}{z-x}\big(h(z)-h(x)\big) \geq h(y)-h(x),$$

which is the first desired inequality. For the second one, it suffices to use the same arguments with $\mu := \frac{z-y}{z-x} \in (0,1)$ and the identity $y = \mu x + (1-\mu)z$.

2)c) Deduce that the following two quantities are well-defined (notice that we allow L here to take the value $+\infty$)

$$L := \lim_{x\to+\infty} \frac{h(x)}{x}, \text{ and } \ell := \lim_{x\to 0} \frac{h(x)-h(0)}{x},$$

and then that for any $0 \leq x < y$

$$\ell \leq \frac{h(y)-h(x)}{y-x} \leq L.$$

Take $x = 0$ in the previous question. We have then that for any $0 < y < z$

$$\frac{h(y)-h(0)}{y} \leq \frac{h(z)-h(0)}{z}.$$

This proves that the map $x \longmapsto \frac{h(x)-h(0)}{x}$ is non–decreasing on $\mathbb{R}_+^\star$, and thus admits a (possibly infinite) limit L as x goes to $+\infty$. Since $\lim_{x\to+\infty} h(0)/x = 0$, we also have

$$\lim_{x\to+\infty} \frac{h(x)}{x} = L.$$

The monotony of $x \longmapsto \frac{h(x)-h(0)}{x}$ also implies the existence of ℓ which is then obviously finite. It then suffices to notice, again by monotony, that for any $0 < x < y$

$$\ell \leq \frac{h(x)-h(0)}{x} \leq \frac{h(y)-h(x)}{y-x} \leq \lim_{z\to+\infty} \left\{\frac{h(z)-h(x)}{z-x}\right\} = L.$$

2)d) Show that for any $t \in \{0, \dots, T\}$ the map $x \longmapsto v(t,x)$ satisfies the same inequalities as h in 2)c), and then that

$$\ell \leq \varphi(t,x) \leq L,\ (t,x) \in \{0, \dots, T-1\} \times \mathbb{R}_+.$$

What can you deduce for European Call and Put options?

We argue by backward induction. For $t = T$ the result is true since h is convex. Let us assume that the result is true for some $t+1 \in \{1, \dots, T\}$. We have for any $0 < x < y$, after some easy manipulations

$$\frac{v(t,y)-v(t,x)}{y-x} = \frac{1}{R}\bigg(uq\frac{v(t+1,uy)-v(t+1,ux)}{u(y-x)} + d(1-q)\frac{v(t+1,dy)-v(t+1,dx)}{d(y-x)}\bigg).$$

By the induction hypothesis, we know that

$$\ell \leq \frac{v(t+1,uy)-v(t+1,ux)}{u(y-x)} \leq L,\ \ell \leq \frac{v(t+1,dy)-v(t+1,dx)}{d(y-x)} \leq L,$$

so that we deduce

$$\ell = \frac{\ell}{R}\big(uq + d(1-q)\big) \leq \frac{v(t,y)-v(t,x)}{y-x} \leq \frac{L}{R}\big(uq + d(1-q)\big) = L,$$

proving thus the first result. Next we have

$$\varphi(t,x) = \frac{v(t+1,ux)-v(t+1,dx)}{(u-d)x},$$

so that φ automatically takes values in $[\ell, L]$.

The previous result shows that for options with convex payoffs, the number of risky assets held in a replicating portfolios is necessarily bounded between ℓ and L which depend solely on the payoff h. For European Call options it is immediate that $L = 1$ and $\ell = 0$, and for Put options that $L = 0$ and $\ell = -1$, which gives us even more information than 1)d), from which we would simply have obtained the signs, and not the upper bound 1 for Call options and the lower bound -1 for Put options.

3)a) Let us define for any $0 \leq a \leq A \leq +\infty$ the set

$$\mathcal{E}_{a,A} := \bigg\{w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \forall (x,y) \in \mathbb{R}_+^2,\ x \neq y,\ \text{we have } a \leq \frac{w(y)-w(x)}{y-x} \leq A\bigg\}.$$

Show that for any $\lambda \in [0,1]$, and for any $(\alpha, \beta) \in (0, +\infty)^2$, the transformation $\Theta_{\lambda,\alpha,\beta}$ defined on $\mathcal{E}_{a,A}$ by

$$\Theta_{\lambda,\alpha,\beta}(w)(x) := \frac{\lambda w(\alpha x) + (1-\lambda) w(\beta x)}{\lambda\alpha + (1-\lambda)\beta},\ x \in \mathbb{R}_+,\ w \in \mathcal{E}_{a,A},$$

is an homomorphism (that is to say that the codomain of $\Theta_{\lambda,\alpha,\beta}$ is $\mathcal{E}_{a,A}$).

We have

$$\frac{\Theta_{\lambda,\alpha,\beta}(w)(y) - \Theta_{\lambda,\alpha,\beta}(w)(x)}{y-x} = \frac{\lambda(w(\alpha y) - w(\alpha x)) + (1-\lambda)(w(\beta y) - w(\beta x))}{(\lambda\alpha + (1-\lambda)\beta)(y-x)}.$$

Thus

$$a = \frac{\lambda\alpha a + (1-\lambda)\beta a}{\lambda\alpha + (1-\lambda)\beta} \leq \frac{\Theta_{\lambda,\alpha,\beta}(w)(y) - \Theta_{\lambda,\alpha,\beta}(w)(x)}{y-x} \leq \frac{\lambda\alpha A + (1-\lambda)\beta A}{\lambda\alpha + (1-\lambda)\beta} = A,$$

which is the desired result.

3)b) Deduce that if $h \in \mathcal{E}_{a,A}$, then

$$a \leq \varphi(t,x) \leq A,\ (t,x) \in \{0,\dots,T-1\} \times \mathbb{R}_+.$$

This is immediate by applying the previous question with $\alpha := u$, $\beta := d$ and $\lambda = q$, which proves by backward induction that for any $t \in \{0,\dots,T\}$, $x \longmapsto v(t,x)$ belongs to $\mathcal{E}_{a,A}$, and thus that φ satisfies the required inequalities.

3)c) Show, using an example, that the result of 3)c) for the replicating strategy is more general than the result of 2)c).

We of course have that whenever h is convex, it belongs to $\mathcal{E}_{\ell,L}$ with the notations of 2)c). However, there are non-convex option payoffs belonging to $\mathcal{E}_{a,A}$ for appropriate choices of a and A. For instance, it is easily seen that any continuous h which has a right-derivative on $\mathbb{R}_+^\star$ taking values in $[a,A]$ belongs to $\mathcal{E}_{a,A}$. This applies to Call spread options with payoffs of the form

$$h(x) = (x-K_1)^+ - (x-K_2)^+,\ x \geq 0,\ 0 \leq K_1 < K_2.$$

4)a) We now consider an American option with maturity T and payoff $h(S_t)$ when it is exercised at time $t \in \{0,\dots,T\}$. You will admit that if p_t is the value of this option at time t, then p_t satisfies the following backward induction (where $\mathbb{Q}$ is the only risk-neutral measure on the market)

$$p_T(\omega) = h\big(S_T(\omega)\big),\ p_t(\omega) = \max\left\{h\big(S_t(\omega)\big), \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[p_{t+1}|\mathcal{F}_t](\omega)\right\},\ (t,\omega) \in \{0,\dots,T-1\} \times \Omega.$$

How can you interpret this formula?

You will also admit that the replicating strategy for such an American option can be obtained, *mutatis mutandis*, with the same recursive formula as for European options. Deduce then that, as in the European option case, we can find a map $v^a : \{0,\dots,T\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$p_t(\omega) = v^a\big(t, S_t(\omega)\big),\ (t,\omega) \in \{0,\dots,T\} \times \Omega,$$

and if $(x,\Delta) \in \mathbb{R}_+ \times \mathcal{A}(\mathbb{R})$ is a replicating strategy for the American option, we can find a map $\varphi^a : \{0,\dots,T-1\} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\Delta_t(\omega) = \varphi^a\big(t, S_t(\omega)\big),\ (t,\omega) \in \{0,\dots,T-1\} \times \Omega.$$

The backward recurrence formula here takes into account the fact that each $t \in \{0,\dots,T\}$, the holder of the American option has to choose whether he wants to exercise it immediately or not. If he does, he receives the amount $h(S_t)$, and if not, at least between t and $t+1$, it is as if he was holding a European option. The value at time t of the American option should therefore be given by the maximum between the values of each of these alternatives. We can then argue as in the European case and get that

$$v^a(T,x) = h(x),\ v^a(t,x) = \max\left\{h(x), \frac{1}{R}\big(qv^a(t+1,ux) + (1-q)v^a(t+1,dx)\big)\right\},\ (t,x) \in \{0,\dots,T-1\} \times \mathbb{R}_+,$$

and

$$\varphi^a(t,x) = \frac{v^a(t+1,ux) - v^a(t+1,dx)}{(u-d)x},$$

4)b) Answer once more questions 1)d) and 2)a) in this context.

It is clear that the maximum between two non-decreasing (resp. non-increasing) functions is itself non–decreasing (resp. non-increasing), and that this is also true for the maximum between two convex functions. This means that we can argue again by backward induction to deduce that whenever h is monotone, $v^a(t,\cdot)$ has the same monotonicity for any $t \in \{0,\dots,T\}$, and that when h is convex, so is $v^a(t,\cdot)$ for any $t \in \{0,\dots,T\}$. The conclusion on the sign of φ^a is the same as well.

4)c) Assume that h is convex, and prove, with the same notations as in 2), that

$$\big|\varphi^a(t,x)\big| \leq \max\big\{|\ell|, |L|\big\},\ (t,x) \in \{0,\dots,T-1\} \times \mathbb{R}_+,$$

The main ingredient in this proof is the following inequality

$$\big|\max\{x,y\} - \max\{w,z\}\big| \leq \max\big\{|x-w|, |y-z|\big\},\ \forall (x,y,w,z) \in \mathbb{R}^4,$$

which can be checked directly. Let us then argue by backward induction. Using 2)c) we know that for $t=T$ and any $0<x<y$

$$\left|\frac{v^a(T,y)-v^a(T,x)}{y-x}\right| = \left|\frac{h(y)-h(x)}{y-x}\right| \leq \max\big\{|\ell|, |L|\big\}.$$

Assume that this is still true for some $t+1 \in \{1,\dots,T\}$, then for any $0<x<y$

$$\begin{aligned}
&\left|\frac{v^a(t,y)-v^a(t,x)}{y-x}\right| \\
&\leq \max\left\{\frac{|h(y)-h(x)|}{y-x},\ \frac{\big|q(v^a(t+1,uy)-v^a(t+1,ux))+(1-q)(v^a(t+1,dy)-v^a(t+1,dx))\big|}{R(y-x)}\right\} \\
&\leq \max\left\{\frac{|h(y)-h(x)|}{y-x},\ qu\frac{|v^a(t+1,uy)-v^a(t+1,ux)|}{Ru(y-x)}+(1-q)d\frac{|v^a(t+1,dy)-v^a(t+1,dx)|}{Rd(y-x)}\right\} \\
&\leq \max\left\{\frac{|h(y)-h(x)|}{y-x},\ \frac{qu}{R}\max\big\{|\ell|,|L|\big\}+(1-q)\frac{d}{R}\max\big\{|\ell|,|L|\big\}\right\} \\
&= \max\left\{\frac{|h(y)-h(x)|}{y-x},\ \max\big\{|\ell|,|L|\big\}\right\} \leq \max\big\{|\ell|,|L|\big\},
\end{aligned}$$

which ends the proof by induction, and implies directly the desired result for φ^a.

4)d) Show that a similar result holds when h is Lipschitz-continuous.

Let us assume that h is ℓ–Lipschitz-continuous for some $\ell>0$, that is to say that

$$|h(x)-h(y)| \leq \ell|x-y|,\ (x,y) \in \mathbb{R}_+^2.$$

Let us argue by backward induction that for any $t \in \{0,\dots,T\}$, $x \longmapsto v^a(t,x)$ is also ℓ–Lipschitz-continuous. For $t=T$ the result is immediate, so let us assume that it holds for some $t+1 \in \{1,\dots,T\}$. We have that for any $(x,y) \in \mathbb{R}_+^2$

$$\begin{aligned}
&\big|v^a(t,x)-v^a(t,y)\big| \\
&\leq \max\left\{|h(x)-h(y)|,\ \frac{1}{R}\big|q(v^a(t+1,ux)-v^a(t+1,uy))+(1-q)(v^a(t+1,dx)-v^a(t+1,dy))\big|\right\} \\
&\leq \max\left\{\ell|x-y|,\ \frac{1}{R}\big(qu\ell|x-y|+(1-q)d\ell|x-y|\right\} = \ell|x-y|.
\end{aligned}$$

We deduce that $|\varphi| \leq \ell$.

4)e) Assume now that $h \in \mathcal{E}_{a,A}$. Does the result of 3)b) extend to the current context?

No, since the result of 3)b) was fundamentally based on the linearity of the relationship between $v(t,\cdot)$ and $v(t+1,\cdot)$, while the relationship between $v^a(t,\cdot)$ and $v^a(t+1,\cdot)$ is now non-linear.

4)f) (Optional). Explain how you would extend the results of 4)a)–4)e) to an American option whose payoff at time $t \in \{0, \dots, T\}$ is now of the form $h(t, S_t)$, for some map $h : \{0, \dots, T\} \times \mathbb{R}_+$.

We now have an American payoff which also depends on the current time. It should be obvious that we still have the formulae for $(t, x) \in \{0, \dots, T-1\} \times \mathbb{R}_+$

$$v^a(T, x) = h(T, x),\ v^a(t, x) = \max\bigg\{h(t, x), \frac{1}{R}\big(qv^a(t+1, ux) + (1-q)v^a(t+1, dx)\big)\bigg\},$$

$$\varphi^a(t, x) = \frac{v^a(t+1, ux) - v^a(t+1, dx)}{(u-d)x}.$$

Hence, whenever for any $t \in \{0, \dots, T\}$, $h(t, \cdot)$ is non-decreasing (resp. non-increasing, convex), so is v^a, and the results of 4)b) still hold.

Next, whenever for any $t \in \{0, \dots, T\}$, $h(t, \cdot)$ is convex, we can denote

$$L(t) := \lim_{x \to +\infty} \frac{h(t, x)}{x},\ \ell(t) := \lim_{x \to 0} \frac{h(t, x) - h(t, 0)}{x},\ L := \max_{t \in \{0, \dots, T\}} L(t),\ \ell := \min_{t \in \{0, \dots, T\}} \ell(t),$$

and we still have that for any $(x, y) \in \mathbb{R}_+^2$ such that $x \neq y$, and for any $t \in \{0, \dots, T\}$

$$\left|\frac{h(t, y) - h(t, x)}{y - x}\right| \leq \max\big\{|L|, |\ell|\big\}.$$

Hence, the exact same arguments as in 4)d) show that $|\varphi^a| \leq \max\big\{|L|, |\ell|\big\}$.

Similarly, if for any $t \in \{0, \dots, T\}$, h is $k(t)$–Lipschitz continuous, then denoting

$$k := \max_{t \in \{0, \dots, T\}} k(t),$$

we can show that $v^a(t, \cdot)$ is k–Lipschitz-continuous, and thus that $|\varphi^a| \leq k$.

Sharpness of call options bounds

The goal of this exercise is to exhibit a financial market in which the bounds

$$(S_t - KB(t, T))^+ \leq C_t(T, K; S) \leq S_t, \tag{0.1}$$

are attained. We thus fix a measurable space $(\Omega, \mathcal{F})$ defined as follows: $\Omega := (0, +\infty)$, and $\mathcal{F}$ is the Borel–σ-algebra on Ω. We let X be the canonical map on Ω, that is

$$X(\omega) = \omega,\ \omega \in \Omega,$$

and we take a probability $\mathbb{P}$ measure on $(\Omega, \mathcal{F})$ making X into a standard log-normal random variable (that is $\log(X)$ has a standard Gaussian distribution).

The model has $T = d = 1$, and we take $\mathcal{F}_0$ trivial, as well as $\mathcal{F}_1 := \mathcal{F}$. The asset prices are given, for some $r \geq 0$, by

$$S_0^0 = 1,\ S_1^0 = \mathrm{e}^r,\ S_0 = 1,\ S_1 = X.$$

1)a) Show that $\mathcal{F}_1 = \mathcal{F} = \sigma(X) = \sigma(S_1)$.

Obviously, the only equality needed to prove here is $\mathcal{F} = \sigma(X)$. Let us go back to definitions. The Borel–σ-algebra on $\Omega = (0, +\infty)$ is the σ-algebra generated by, for instance, open intervals in $(0, +\infty)$. Since X is $(0, +\infty)$-valued, and since $\mathcal{F}$ is the Borel–σ-algebra on $(0, +\infty)$, we have by definition that

$$\sigma(X) = \big\{X^{-1}(A) : A \in \mathcal{F}\big\}.$$

Take an open interval $(a, b) \subset (0, +\infty)$. It is immediate that $X^{-1}\big((a, b)\big) = (a, b)$, and therefore $\sigma(X)$ contains all open intervals of $(0, +\infty)$ proving that $\mathcal{F} \subset \sigma(X)$. Since the converse inclusion is immediate, this proves the result.

1)b) Show that the probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ with density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \exp\bigg(\Big(r-\frac{1}{2}\Big)\log(X) - \frac{1}{2}\Big(r-\frac{1}{2}\Big)^2\bigg),$$

is well-defined and is a risk-neutral measure.

The density is positive, and we have, since $\log(X)$ has a Gaussian distribution with mean 0 and variance 1

$$\mathbb{E}^{\mathbb{P}}\bigg[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg] = \exp\bigg(-\frac{1}{2}\Big(r-\frac{1}{2}\Big)^2\bigg)\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\Big(r-\frac{1}{2}\Big)\log(X)\bigg)\bigg] = 1.$$

This shows that $\mathbb{Q}$ is well-defined. Besides

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r}S_1\big] = \mathrm{e}^{-r}\mathbb{E}^{\mathbb{P}}\bigg[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}X\bigg] &= \exp\bigg(-r-\frac{1}{2}\Big(r-\frac{1}{2}\Big)^2\bigg)\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\Big(r+\frac{1}{2}\Big)\log(X)\bigg)\bigg]\\ &= \exp\bigg(-r-\frac{1}{2}\Big(r-\frac{1}{2}\Big)^2+\frac{1}{2}\Big(r+\frac{1}{2}\Big)^2\bigg)\\ &= 1 = S_0,\end{aligned}$$

proving thus that $\widetilde{S}$ is an $(\mathbb{F},\mathbb{Q})$-martingale (integrability here is obvious). Since $\mathbb{Q}$ is by definition equivalent to $\mathbb{P}$ (recall that the density is positive), we indeed have $\mathbb{Q}\in\mathcal{M}(S)$.

1)c) Show that the market is however incomplete by constructing a non-replicable payoff.

Let us first try and see what replicable payoffs must look like in this market. We fix therefore some $\mathcal{F}_1$-measurable random variable ξ. Notice that since $\mathcal{F}_1=\sigma(S_1)$, the Doob–Dynkin lemma implies that there must exist a measurable map $h:(0,+\infty)\longrightarrow\mathbb{R}$ such that $\xi=h(S_1)$. If that payoff is replicable, then we must be able to find $(x,\Delta)\in\mathbb{R}^2$ such that

$$x+\Delta(\mathrm{e}^{-r}S_1-1) = \widetilde{X}_1^{x,\Delta} = \mathrm{e}^{-r}h(S_1),\ \mathbb{P}\text{-a.s.}$$

Assume that we have chosen a payoff ξ such that h is a continuous function. Since the support of the distribution of S_1 under $\mathbb{P}$ is $[0,+\infty)$, by continuity of both sides of the previous equation with respect to S_1, the equality can hold if and only if

$$x+\Delta(\mathrm{e}^{-r}y-1) = \mathrm{e}^{-r}h(y),\ \forall y\in[0,+\infty).$$

This implies that if a payoff of the form $h(S_1)$ with h continuous is replicable, then h has to be affine on $[0,+\infty)$. Conversely, it is obvious that if $h:(0,+\infty)\longrightarrow\mathbb{R}$ is continuous and affine on $(0,+\infty)$, then the payoff $h(S_1)$ is replicable. Therefore, any continuous payoff which is not affine on $(0,+\infty)$ cannot be replicated in this market. A typical example would be a call or a put option with strike $K>0$, whose respective payoffs $(y-K)^+$ and $(K-y)^+$ are non-linear on $(0,+\infty)$. Therefore the market is incomplete.

2) Let $\mathcal{P}$ be the set of all probability measures on $(\Omega,\mathcal{F})$. We now define a subset $\mathcal{P}_{\text{bin}}$ of $\mathcal{P}$, consisting of all martingale measures for S which in addition make the market into a binomial one, that is to say[1]

$$\mathcal{P}_{\text{bin}} := \Big\{\Pi\in\mathcal{P}:\Pi\circ(S)^{-1}\text{ has mass in two points, and }\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}S_1\big]=1\Big\}.$$

2)a) Are elements of $\mathcal{P}_{\text{bin}}$ risk-neutral measures? Why?

Under any measure $\Pi\in\mathcal{P}_{\text{bin}}$, the distribution of S_1 is discrete, while it is continuous under $\mathbb{P}$. This means that measures in $\mathcal{P}_{\text{bin}}$ cannot be equivalent to $\mathbb{P}$, and thus are not risk-neutral measures, despite making the risky asset S into a martingale.

[1]The notation $\Pi\circ(S)^{-1}$ represents the distribution of S under Π. In more measure-theoretic terms, this is simply the image measure of Π through the measurable map $S:\Omega\longrightarrow(0,+\infty)$.

2)b) Fix some $0 < d < \mathrm{e}^r < u$. Construct a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{M}(S)$ (which thus must be equivalent to $\mathbb{P}$), but which converges weakly to some $\Pi \in \mathcal{P}_{\text{bin}}$ which has mass at u and d.

Notice first that defining here the sequence $(\Pi_n)_{n\in\mathbb{N}}$ or the sequence $(\mu_n)_{n\in\mathbb{N}}$, where

$$\mu_n := \Pi_n \circ (S_1)^{-1},\ n \in \mathbb{N},$$

is completely equivalent. Indeed, for any $n \in \mathbb{N}$, one can obviously construct μ_n from Π_n, and conversely, being given μ_n, we can re-construct Π_n by taking it as the pull-back of μ_n by S_1 (since S_1 is obviously here a bijection from $(0,+\infty)$ to $(0,+\infty)$).

We will therefore construct $(\mu_n)_{n\in\mathbb{N}}$. First, since the $(\Pi_n)_{n\in\mathbb{N}}$ have to be equivalent to $\mathbb{P}$, and since $\mathbb{P}\circ(S_1)^{-1}$ is equivalent to the Lebesgue measure on $(0,+\infty)$, all the $(\mu_n)_{n\in\mathbb{N}}$ must be equivalent to Lebesgue measure on $(0,+\infty)$. Let us denote for any $n \in \mathbb{N}$ by $f_n : (0,+\infty) \longrightarrow (0,+\infty)$ the density of μ_n with respect to Lebesgue measure on $(0,+\infty)$.

Being given u and d as in the question, the idea is that we want to make the densities $(f_n)_{n\in\mathbb{N}}$ 'look like' the map $f : (0,+\infty) \longrightarrow (0,+\infty)$ where

$$f(x) = \frac{\mathrm{e}^r - d}{u-d}\mathbf{1}_{\{x=u\}} + \frac{u-\mathrm{e}^r}{u-d}\mathbf{1}_{\{x=d\}},$$

since this is the 'density' of the measure

$$\mu := \frac{\mathrm{e}^r - d}{u-d}\delta_{\{u\}} + \frac{u-\mathrm{e}^r}{u-d}\delta\{d\},$$

which is the only possible distribution for S^1 concentrated on u and d, under a measure in $\mathcal{P}_{\text{bin}}$ (because of the required martingale property).

It is therefore natural to define for any $n \in \mathbb{N}$

$$f_n(x) := a_n\mathbf{1}_{\{x\in[u,u+1/(n+1))\}} + b_n\mathbf{1}_{\{x\in[d,d+1/(n+1))\}},\ x \in (0,+\infty),$$

where the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ need to be determined so as to make sure that the $(f_n)_{n\in\mathbb{N}}$ are densities, and that S is an $(\mathbb{F},\Pi_n)$-martingale for any $n\in\mathbb{N}$. Since $d<u$, we can assume without loss of generality that n is large enough so that intervals $[d,d+1/(n+1))$ and $[u,u+1/(n+1))$ are disjoint. Both these conditions then translate (notice that again integrability is immediate here) to needing for any $n\in\mathbb{N}$ that $a_n>0$, $b_n>0$ and

$$\begin{aligned}
1 &= \int_{(0,+\infty)} f_n(x)\mathrm{d}x = \frac{a_n+b_n}{n+1},\\
1 &= \mathrm{e}^{-r}\int_{(0,+\infty)} x f_n(x)\mathrm{d}x = \frac{\mathrm{e}^{-r}}{2}\left(a_n\frac{1+2u(n+1)}{(n+1)^2} + b_n\frac{1+2d(n+1)}{(n+1)^2}\right).
\end{aligned}$$

This leads immediately to

$$a_n = (n+1)\frac{\mathrm{e}^r-d}{u-d} - \frac{1}{2(u-d)},\ b_n = (n+1)\frac{u-\mathrm{e}^r}{u-d} + \frac{1}{2(u-d)},$$

which are indeed positive provided that n is large enough. Now fix some bounded continuous map $\varphi : (0,+\infty) \longrightarrow \mathbb{R}$. We have

$$\begin{aligned}
\int_{(0,+\infty)} \varphi(x)\mathrm{d}\mu_n(x) = &\left((n+1)\frac{u-\mathrm{e}^r}{u-d} + \frac{1}{2(u-d)}\right)\int_d^{d+\frac{1}{n+1}}\varphi(x)\mathrm{d}x\\
&+\left((n+1)\frac{\mathrm{e}^r-d}{u-d} - \frac{1}{2(u-d)}\right)\int_u^{u+\frac{1}{n+1}}\varphi(x)\mathrm{d}x
\end{aligned}$$

Since φ is continuous, the Lebesgue differentiation theorem ensures that

$$\int_{(0,+\infty)} \varphi(x)\mathrm{d}\mu_n(x) \underset{n\to+\infty}{\longrightarrow} \frac{\mathrm{e}^r - d}{u-d}\varphi(u) + \frac{u-\mathrm{e}^r}{u-d}\varphi(d) = \int_{(0,+\infty)} \varphi(x)\mathrm{d}\mu(x),$$

proving thus the weak convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$ to μ.

2)c) Define now the set

$$\mathfrak{P}_{\mathrm{bin}} := \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1-K)^+\big] : \Pi \in \mathcal{P}_{\mathrm{bin}}\Big\}.$$

Show that

$$\mathfrak{P}_{\mathrm{bin}} \subset \Big[-p\big(-(S_1-K)^+\big), p\big((S_1-K)^+\big)\Big].$$

Hint: it could be useful to use convex combinations of $\mathbb{Q}$ and elements of the sequences $(\Pi_n)_{n\in\mathbb{N}}$ from 2)b).

Start by noticing that

$$\mathfrak{P}_{\mathrm{bin}} = \bigg\{\frac{\mathrm{e}^r-d}{u-d}\mathrm{e}^{-r}(u-K)^+ + \frac{u-\mathrm{e}^r}{u-d}\mathrm{e}^{-r}(d-K)^+ : 0<d<\mathrm{e}^r<u\bigg\}.$$

Fix thus some $0<d<\mathrm{e}^r<u$, and let $n\in\mathbb{N}$ be large enough so that the construction of the previous question can be carried out for the pair (d,u), and define the measure

$$\mathbb{Q}_n := \frac{1}{n+1}\mathbb{Q} + \frac{n}{n+1}\Pi_n.$$

It is immediate that $\mathbb{Q}_n \in \mathcal{M}(S)$, as a convex combination of measures in $\mathcal{M}(S)$. As such, $\mathbb{E}^{\mathbb{Q}_n}[\mathrm{e}^{-r}(S_1-K)^+]$ is a viable price for the call option with strike K, and thus

$$\mathbb{E}^{\mathbb{Q}_n}\big[\mathrm{e}^{-r}(S_1-K)^+\big] \in \Big(-p\big(-(S_1-K)^+\big), p\big((S_1-K)^+\big)\Big). \tag{0.2}$$

In addition, we have by definition

$$\mathbb{E}^{\mathbb{Q}_n}\big[\mathrm{e}^{-r}(S_1-K)^+\big] = \frac{1}{n+1}\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r}(S_1-K)^+\big] + \frac{n}{n+1}\mathbb{E}^{\Pi_n}\big[\mathrm{e}^{-r}(S_1-K)^+\big].$$

The first term on the right-hand side goes to 0 as n goes to $+\infty$. As for the second one, we have using the fact that the support of μ_n is included in $[0,u+1]$

$$\mathbb{E}^{\Pi_n}\big[\mathrm{e}^{-r}(S_1-K)^+\big] = \int_{(0,+\infty)} \mathrm{e}^{-r}(x-K)^+\mathrm{d}\mu_n(x) = \int_0^{u+1} \mathrm{e}^{-r}(x-K)^+\mathrm{d}\mu_n(x).$$

Since $x \longmapsto (x-K)^+$ is continuous and bounded on the compact $[0,u+1]$, by weak convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$ we deduce that

$$\mathbb{E}^{\mathbb{Q}_n}\big[\mathrm{e}^{-r}(S_1-K)^+\big] \underset{n\to+\infty}{\longrightarrow} \frac{\mathrm{e}^r-d}{u-d}\mathrm{e}^{-r}(u-K)^+ + \frac{u-\mathrm{e}^r}{u-d}\mathrm{e}^{-r}(d-K)^+.$$

By (0.2), this implies that

$$\frac{\mathrm{e}^r-d}{u-d}\mathrm{e}^{-r}(u-K)^+ + \frac{u-\mathrm{e}^r}{u-d}\mathrm{e}^{-r}(d-K)^+ \in \Big[-p\big(-(S_1-K)^+\big), p\big((S_1-K)^+\big)\Big],$$

and proves the desired result by arbitrariness of u and d.

2)d) Show that

$$\sup_{\Pi\in\mathcal{P}_{\mathrm{bin}}} \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1-K)^+\big]\Big\} = 1,\ \inf_{\Pi\in\mathcal{P}_{\mathrm{bin}}} \Big\{\mathbb{E}^{\Pi}\big[\mathrm{e}^{-r}(S_1-K)^+\big]\Big\} = (1-K\mathrm{e}^{-r})^+,$$

and deduce that the universal bounds in (0.1) (for $t=0$) are attained in this market.

First, one should realise that the universal bounds show that

$$\sup_{\Pi\in\mathcal{P}_{\rm bin}}\Big\{\mathbb{E}^\Pi\big[\mathrm{e}^{-r}(S_1-K)^+\big]\Big\}\leq 1,\ \inf_{\Pi\in\mathcal{P}_{\rm bin}}\Big\{\mathbb{E}^\Pi\big[\mathrm{e}^{-r}(S_1-K)^+\big]\Big\}\geq (1-K\mathrm{e}^{-r})^+.$$

It thus suffices to prove here the reverse inequalities.

Define the following map on $A:=\big\{(u,d)\in(0,+\infty)^2:d<\mathrm{e}^r<u\big\}$

$$g(d,u):=\frac{\mathrm{e}^r-d}{u-d}\mathrm{e}^{-r}(u-K)^++\frac{u-\mathrm{e}^R}{u-d}\mathrm{e}^{-r}(d-K)^+,\ (d,u)\in A,$$

We have directly

$$\frac{\partial g}{\partial u}(d,u)=\mathbf{1}_{\{u>K\}}\mathrm{e}^{-r}\frac{\mathrm{e}^r-d}{(u-d)^2}(K-d)^+,\ \frac{\partial g}{\partial d}(d,u)=-\mathbf{1}_{\{d<K\}}\mathrm{e}^{-r}\frac{\mathrm{e}^r-d}{(u-d)^2}(u-K)^+,$$

proving that g is non-decreasing with respect to u, and non-increasing with respect to d. As such, we can attain the supremum in the question by letting u go appropriately to $+\infty$ and d to 0. For the supremum, this is a bit more subtle due to the indicator functions above, and this will mostly depend on whether K is above e^r or not.

Take thus for n large enough $u=n$ and $d=n^{-1}$. Increasing n if necessary, we have

$$g\big(n^{-1},n\big)=\frac{n\mathrm{e}^r-1}{n(n-1)}\mathrm{e}^{-r}(n-K)\underset{n\to+\infty}{\longrightarrow}1,$$

which indeed attains the upper bound.

For the lower bound, we need to distinguish two cases. First if $K\geq\mathrm{e}^r$, then we are trying to reach the value 0. For this, we can take $d=n^{-1}$ and $u=K+n^{-1}$. We then have for n large enough

$$g\big(n^{-1},K+n^{-1}\big)=\frac{n\mathrm{e}^r-1}{kn^2}\mathrm{e}^{-r}\underset{n\to+\infty}{\longrightarrow}0.$$

In the case $K<\mathrm{e}^r$, take any $u>\mathrm{e}^r$, and any $d\in(K,1)$, leading to

$$g(d,u)=\frac{\mathrm{e}^r-d}{u-d}\mathrm{e}^{-r}(u-K)+\frac{u-\mathrm{e}^R}{u-d}\mathrm{e}^{-r}(d-K)=1-K\mathrm{e}^{-r}.$$

This ends the proof.