Introduction to Mathematical Finance
Dylan Possamaï

Assignment 8 (solutions)

About Asian options

We consider a complete T-period financial market, such that (**NA**) holds. There is a risk-less asset which is for now such that $(1/S^0_t)_{t\in\{0,\dots,T\}}$ is a positive $(\mathbb{F},\mathbb{Q})$–super-martingale, where $\mathbb{Q}$ is the unique risk-neutral measure on this market. There is only one risky asset with price process S.

We fix some $K\geq 0$, and we are interested in a so-called Asian Call option on S, whose payoff at maturity T is given by

$$\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+.$$

We will denote by $C^{\mathrm{as}}_t(T,K;S)$ the value at any time $t\in\{0,\dots,T\}$ of such an option. For notational simplicity, we will also take the convention in the formulae below that $\frac{0}{0}=0$.

1)$a)$ Show that $\mathbb{P}$-a.s.

$$\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+ \leq \frac{1}{T}\sum_{k=1}^T (S_k-K)^+.$$

It suffices to use here Jensen's inequality for the convex map $x\longmapsto x^+$ to get

$$\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+ = \left(\frac{1}{T}\sum_{k=1}^T (S_k - K)\right)^+ \leq \frac{1}{T}\sum_{k=1}^T (S_k-K)^+.$$

1)$b)$ Deduce that

$$C^{\mathrm{as}}_0(T,K;S)\leq \frac{1}{T}\sum_{k=1}^T C_0(k,K;S).$$

Since the market is complete here, we have thanks to the first question and the super-martingale property $1/S^0$

$$\begin{aligned}
C^{\mathrm{as}}_0(T,K;S) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S^0_T}\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+\right] &\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S^0_T}\frac{1}{T}\sum_{k=1}^T (S_k-K)^+\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\sum_{k=1}^T\left((S_k-K)^+\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S^0_T}\middle|\mathcal{F}_k\right]\right)\right]\\
&\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\sum_{k=1}^T\left((S_k-K)^+\frac{1}{S^0_k}\right)\right]\\
&= \frac{1}{T}\sum_{k=1}^T\mathbb{E}^{\mathbb{Q}}\left[\frac{(S_k-K)^+}{S^0_k}\right]\\
&= \frac{1}{T}\sum_{k=1}^T C_0(k,K;S).
\end{aligned}$$

$2)a)$ Show that for any $t \in \{0,\dots,T\}$ and any $s \in \{t,\dots,T\}$

$$\frac{(S_t-K)^+}{S_t^0} \leq \left(\widetilde{S}_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{K}{S_s^0}\middle|\mathcal{F}_t\right]\right)^+, \ \mathbb{P}\text{-a.s.}$$

We have since S^0 is a positive $(\mathbb{F},\mathbb{Q})$–super-martingale

$$\frac{(S_t-K)^+}{S_t^0} = \left(\widetilde{S}_t - \frac{K}{S_t^0}\right)^+ \leq \left(\widetilde{S}_t - K\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_s^0}\middle|\mathcal{F}_t\right]\right)^+.$$

$2)b)$ Deduce using Jensen's inequality for conditional expectations that for any $t \in \{0,\dots,T\}$, with $\mathbb{P}$-probability one, the sequence $(C_t(k,K;S))_{k\in\{t,\dots,T\}}$ is non-decreasing.

Fix some some $k \in \{t,\dots,T\}$ and some $s \in \{k,\dots,T\}$. We will prove that $C_t(k,K;S) \leq C_t(s,K;S)$. We have by the previous question and the fact that $\widetilde{S}$ is an $(\mathbb{F},\mathbb{Q})$-martingale

$$\begin{aligned}\frac{(S_k-K)^+}{S_k^0} &\leq \left(\widetilde{S}_k - \mathbb{E}^{\mathbb{Q}}\left[\frac{K}{S_s^0}\middle|\mathcal{F}_k\right]\right)^+\\ &= \left(\mathbb{E}^{\mathbb{Q}}\left[\widetilde{S}_s\middle|\mathcal{F}_k\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{K}{S_s^0}\middle|\mathcal{F}_k\right]\right)^+\\ &\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{(S_s-K)^+}{S_s^0}\middle|\mathcal{F}_k\right].\end{aligned}$$

Taking conditional expectations on both sides with respect to $\mathcal{F}_t$, we deduce by the tower property fro conditional expectations

$$C_t(k,K;S) \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{(S_s-K)^+}{S_s^0}\middle|\mathcal{F}_t\right] = C_t(s,K;S).$$

$2)c)$ Show that

$$C_0^{\text{as}}(T,K;S) \leq C_0(T,K;S).$$

Using the previous questions, we have

$$C_0^{\text{as}}(T,K;S) \leq \frac{1}{T}\sum_{k=1}^{T} C_0(k,K;S) \leq \frac{1}{T}\sum_{k=1}^{T} C_0(T,K;S) = C_0(T,K;S).$$

The result is intuitive. Indeed, one would expect that it is 'harder' for the average value of the risky asset over $\{1,\dots,T\}$ to remain above the strike K, than it is for the terminal value S_T to be above K. As such we can expect the price of the Asian Call option to be lower than that of the standard European Call option.

3) In this question we will extend the previous results to any time $t \in \{0,\dots,T\}$.

$3)a)$ Show that for any $t \in \{0,\dots,T\}$, we have $\mathbb{P}$-a.s.

$$C_t^{\text{as}}(T,K;S) \leq \frac{t}{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\middle|\mathcal{F}_t\right]\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T} C_t(k,K;S).$$

Notice first that we can rewrite for any $t \in \{1,\dots,T\}$

$$\frac{1}{T}\sum_{k=1}^{T} S_k - K = \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right) + \frac{1}{T}\sum_{k=t+1}^{T}(S_k - K).$$

Since this is a convex combination of $\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K, S_{t+1}-K, S_{t+2}-K,\dots,S_T-K\right)$**, we have by convexity of** $x \longmapsto x^+$

$$\left(\frac{1}{T}\sum_{k=1}^{T} S_k - K\right)^+ \leq \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T}(S_k - K)^+$$

The market being complete, we have

$$\begin{aligned} C_t^{\mathrm{as}}(T,K;S) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\left(\frac{1}{T}\sum_{k=1}^{T} S_k - K\right)^+ \middle| \mathcal{F}_t\right] \\ &\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\left(\frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T}(S_k - K)^+\right)\middle| \mathcal{F}_t\right] \\ &= \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\middle|\mathcal{F}_t\right] + \frac{1}{T}\sum_{k=t+1}^{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}(S_k-K)^+\middle|\mathcal{F}_t\right]. \end{aligned} \tag{0.1}$$

It suffices to conclude to notice that since S^0 **is an** $(\mathbb{F},\mathbb{Q})$**–super-martingale, we have for any** $k \in \{t+1,\dots,T\}$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}(S_k-K)^+\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[S_t^0(S_k-K)^+\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_T^0}\middle|\mathcal{F}_k\right]\middle|\mathcal{F}_t\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_k^0}(S_k-K)^+\middle|\mathcal{F}_t\right] = C_t(k,K;S).$$

3)b) Show that the result in 3)a) is indeed a generalisation of 2)c).

This is clearly the case. since by taking $t=0$ **in the result of** 3)a)**, and using the fact that the European Call prices are non-decreasing with respect to maturity, we recover the result of** 2)c)**.**

4) From now on, and until the end of the problem, we assume that S^0 is deterministic. Prove that we can now write for any $t \in \{0,\dots,T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T,K;S) \leq \frac{t}{T}\frac{S_t^0}{S_T^0}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ + \frac{1}{T}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}C_t(k,K;S).$$

We go back to (0.1) **and use the deterministic nature of** S^0

$$\begin{aligned} C_t^{\mathrm{as}}(T,K;S) &= \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\middle|\mathcal{F}_t\right] + \frac{1}{T}\sum_{k=t+1}^{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}(S_k-K)^+\middle|\mathcal{F}_t\right] \\ &= \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ \frac{S_t^0}{S_T^0} + \frac{1}{T}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_k^0}(S_k-K)^+\middle|\mathcal{F}_t\right] \\ &= \frac{t}{T}\left(\frac{1}{t}\sum_{k=1}^{t} S_k - K\right)^+ \frac{S_t^0}{S_T^0} + \frac{1}{T}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}C_t(k,K;S) \end{aligned}$$

5) Define for any $t \in \{0, \dots, T\}$ the event

$$A(t) := \left\{ \frac{1}{T} \sum_{k=1}^{t} S_k \geq K \right\} \in \mathcal{F}_t.$$

Show that for any $t \in \{0, \dots, T\}$, $\mathbb{P}$-a.s.

$$\mathbf{1}_{A(t)} C_t^{\mathrm{as}}(T, K; S) = \mathbf{1}_{A(t)} \left(\frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{t} S_k - K \right) + \frac{S_t}{T} \sum_{k=t+1}^{T} \frac{S_k^0}{S_T^0} \right).$$

Notice that since S is positive, we have

$$\mathbf{1}_{A(t)} \left(\frac{1}{T} \sum_{k=1}^{T} S_k - K \right)^+ = \mathbf{1}_{A(t)} \left(\frac{1}{T} \sum_{k=1}^{T} S_k - K \right).$$

We thus deduce

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{A(t)} \frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{T} S_k - K \right)^+ \middle| \mathcal{F}_t \right] &= \mathbf{1}_{A(t)} \mathbb{E}^{\mathbb{Q}} \left[\frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{T} S_k - K \right) \middle| \mathcal{F}_t \right] \\
&= \mathbf{1}_{A(t)} \left(\frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{t} S_k - K \right) + \frac{1}{T} \sum_{k=t+1}^{T} \mathbb{E}^{\mathbb{Q}} \left[\frac{S_t^0}{S_T^0} S_k \middle| \mathcal{F}_t \right] \right) \\
&= \mathbf{1}_{A(t)} \left(\frac{S_t^0}{S_T^0} \left(\frac{1}{T} \sum_{k=1}^{t} S_k - K \right) + \frac{1}{T} \sum_{k=t+1}^{T} \frac{S_k^0}{S_T^0} S_t \right).
\end{aligned}$$

6) Fix now some $T_o \in \{1, \dots, T-1\}$ and consider the option with maturity T and payoff

$$\left(\frac{1}{T - T_o} \sum_{k=T_o+1}^{T} S_k - K \right)^+.$$

We denote by $C^{\mathrm{as}}(T_o, T, K; S)$ the price process of the corresponding option.

6)a) Show that for any $t \in \{T_o, \dots, T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T_o, T, K; S) \leq \frac{t - T_o}{T - T_o} \frac{S_t^0}{S_T^0} \left(\frac{1}{t - T_o} \sum_{k=T_o+1}^{t} S_k - K \right)^+ + \frac{1}{T - T_o} \sum_{k=t+1}^{T} \frac{S_k^0}{S_T^0} C_t(k, K; S).$$

Notice first that we can rewrite for any $t \in \{T_o, \dots, T\}$

$$\frac{1}{T - T_o} \sum_{k=T_o+1}^{T} S_k - K = \frac{t - T_o}{T - T_o} \left(\frac{1}{t - T_o} \sum_{k=T_o+1}^{t} S_k - K \right) + \frac{1}{T - T_o} \sum_{k=t+1}^{T} (S_k - K).$$

Since this is a convex combination of $\left(\frac{1}{t - T_o} \sum_{k=1}^{t} S_k - K, S_{t+1} - K, S_{t+2} - K, \dots, S_T - K \right)$, we have by convexity of $x \longmapsto x^+$

$$\left(\frac{1}{T - T_o} \sum_{k=T_o+1}^{T} S_k - K \right)^+ \leq \frac{t - T_o}{T - T_o} \left(\frac{1}{t - T_o} \sum_{k=T_o+1}^{t} S_k - K \right)^+ + \frac{1}{T - T_o} \sum_{k=t+1}^{T} (S_k - K)^+.$$

The market being complete, we have

$$\begin{aligned}
C_t^{\mathrm{as}}(T_o,T,K;S) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\left(\frac{1}{T-T_o}\sum_{k=T_o+1}^{T} S_k - K\right)^+\middle|\mathcal{F}_t\right]\\
&\leq \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0}\left(\frac{t-T_o}{T-T_o}\left(\frac{1}{t-T_o}\sum_{k=T_o+1}^{t} S_k - K\right)^+ + \frac{1}{T-T_o}\sum_{k=t+1}^{T}(S_k-K)^+\right)\middle|\mathcal{F}_t\right]\\
&= \frac{t-T_o}{T-T_o}\left(\frac{1}{t-T_o}\sum_{k=T_o+1}^{t} S_k - K\right)^+\frac{S_t^0}{S_T^0} + \frac{1}{T-T_o}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_k^0}(S_k-K)^+\middle|\mathcal{F}_t\right]\\
&= \frac{t-T_o}{T-T_o}\left(\frac{1}{t-T_o}\sum_{k=T_o+1}^{t} S_k - K\right)^+\frac{S_t^0}{S_T^0} + \frac{1}{T-T_o}\sum_{k=t+1}^{T}\frac{S_k^0}{S_T^0}C_t(k,K;S).
\end{aligned}$$

$6)b)$ Show that for any $t\in\{T_o,\dots,T\}$, $\mathbb{P}$-a.s.

$$C_t^{\mathrm{as}}(T_o,T,K;S) \leq \frac{1}{T-T_o}\sum_{k=T_o+1}^{T}\frac{S_k^0}{S_T^0}C_t(k,K;S).$$

Notice that since the market is complete, the discounted value of any positive option price is an $(\mathbb{F},\mathbb{Q})$-martingale. Indeed, since the options can be replicated by a self-financing portfolio, we know that their discounted values are $(\mathbb{F},\mathbb{Q})$-local martingales. Since in addition their values are positive, it is immediate that the negative part of their values at T is $\mathbb{Q}$-integrable, and there are therefore $(\mathbb{F},\mathbb{Q})$-martingales. As such, we have for any $t\in\{T_o,\dots,T\}$, using the result of $6)a)$

$$\begin{aligned}
\frac{1}{S_t^0}C_t^{\mathrm{as}}(T_o,T,K;S) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_{T_o}^0}C_{T_o}^{\mathrm{as}}(T_o,T,K;S)\middle|\mathcal{F}_t\right]\\
&\leq \frac{1}{T-T_o}\sum_{k=T_o+1}^{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_k^0}{S_{T_o}^0S_T^0}C_{T_o}(k,K;S)\middle|\mathcal{F}_t\right]\\
&= \frac{1}{T-T_o}\sum_{k=T_o+1}^{T}\frac{S_k^0}{S_T^0}\frac{C_t(k,K;S)}{S_t^0},
\end{aligned}$$

where we used the fact that for any $k\in\{T_o+1,\dots,T\}$, $\left(\frac{C_t(k,K;S)}{S_t^0}\right)_{t\in\{T_o,\dots,k\}}$ is an $(\mathbb{F},\mathbb{Q})$-martingale.

7) We now specialise the discussion to a more specific model and assume that for some $R>0$

$$S_t^0 = R^t,\ t\in\{0,\dots,T\},$$

and that the risky asset satisfies that $S_0>0$ is given and

$$S_{t+1} = Y_{t+1}S_t,\ t\in\{0,\dots,T-1\},$$

where the $(Y_i)_{i\in\{1,\dots,T\}}$ are i.i.d. random variables under $\mathbb{Q}$, taking values in $(0,+\infty)$.

$7)a)$ Explain first why

$$\mathbb{E}^{\mathbb{Q}}[Y_i] = R,\ i\in\{1,\dots,T\}.$$

Since $\mathbb{Q}$ is a risk-neutral measure, $\widetilde{S}$ must be an $(\mathbb{F},\mathbb{Q})$-martingale. This means here that for any $t\in\{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t+1}}{R^{t+1}}\middle|\mathcal{F}_t\right] = \frac{S_t}{R^t},\ \textbf{which implies } \mathbb{E}^{\mathbb{Q}}[Y_{t+1}|\mathcal{F}_t] = R,$$

which is the desired result by taking expectations again.

7)b) Prove then that, $\mathbb{P}$-a.s.

$$\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+ \geq \left(S_0 \exp\left(\sum_{k=1}^T \left(1-\frac{k-1}{T}\right)\log(Y_k)\right) - K\right)^+,$$

and then that

$$C_0^{\mathrm{as}}(T,K;S) \geq \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[\left(S_0 \exp\left(\sum_{k=1}^T \frac{k}{T}\log(Y_k)\right) - K\right)^+\right].$$

We have using Jensen's inequality for the convex function $x \longmapsto \mathrm{e}^x$

$$\frac{1}{T}\sum_{k=1}^T S_k = \frac{S_0}{T}\sum_{k=1}^T \left(\prod_{\ell=1}^k Y_\ell\right) = \frac{S_0}{T}\sum_{k=1}^T \exp\left(\sum_{\ell=1}^k \log(Y_\ell)\right) \leq S_0 \exp\left(\frac{1}{T}\sum_{k=1}^T\sum_{\ell=1}^k \log(Y_\ell)\right)$$
$$= S_0 \exp\left(\frac{1}{T}\sum_{\ell=1}^T (T-\ell+1)\log(Y_\ell)\right),$$

which is the first desired result.

Then, we deduce

$$C_0^{\mathrm{as}}(T,K;S) = \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\sum_{k=1}^T S_k - K\right)^+\right] \geq \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[\left(S_0 \exp\left(\sum_{\ell=1}^T \left(1-\frac{\ell-1}{T}\right)\log(Y_\ell)\right) - K\right)^+\right]$$
$$= \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[\left(S_0 \exp\left(\sum_{k=1}^T \frac{k}{T}\log(Y_{T-k+1})\right) - K\right)^+\right]$$
$$= \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[\left(S_0 \exp\left(\sum_{k=1}^T \frac{k}{T}\log(Y_k)\right) - K\right)^+\right],$$

where in the last equality we used the fact that since the $(Y_i)_{i\in\{1,\dots,T\}}$ are i.i.d. under $\mathbb{Q}$, the $\mathbb{Q}$-distribution of $(Y_1,\dots,Y_T)$ is the same as the $\mathbb{Q}$-distribution of $(Y_T,\dots,Y_1)$.

7)c) Define the following process

$$\overline{S}_t := S_0 \exp\left(\frac{1}{t}\sum_{k=1}^t k\log(Y_k)\right),\ t \in \{0,\dots,T\}.$$

Show that the lower bound obtained in 7)b) can be written formally

$$C_0^{\mathrm{as}}(T,K;S) \geq C_0(T,K;\overline{S}).$$

Can we however say that $\overline{S}$ is the price process of a (fictitious) risky asset in this market?

The first part of the question is immediate, since we have shown

$$C_0^{\mathrm{as}}(T,K;S) \geq \frac{1}{R^T}\mathbb{E}^{\mathbb{Q}}\left[(\overline{S}_T - K)^+\right] = C_0(T,K;\overline{S}).$$

Now for $\overline{S}$ to ba considered as a risky-asset on the market, it's discounted value should be an $(\mathbb{F},\mathbb{Q})$-martingale. But this has absolutely no reason to be the case here. Hence $\overline{S}$ is some sort of pseudo–risky asset.