Assignment 9 (solutions)

About Futures

We consider a complete T-period financial market, such that (**NA**) holds, and we let $\mathbb{Q}$ be the unique risk-neutral measure on this market.

Futures contracts, unlike forward contracts, are marked-to-market, meaning that they receive cash-flows at every trading dates. More precisely, a futures contract is an agreement to purchase an asset at the maturity T, for a pre-specified price, called the *futures price*. This futures price is paid via a sequence of instalments over the contract's life. As with forward contracts, no cash-flow happens at the inception of the contract, supposed to correspond to time 0 here. However, a cash payment is made at every trading date, corresponding to the change in the futures price between this date and the previous trading one. Mathematically, if we define the futures price at time t, for an asset S with maturity T by $G_t(T;S_T)$, then the cash-flows are

$$G_t(T;S_T) - G_{t-1}(T;S_T), \text{ at time } t \in \{1,\dots,T\}.$$

Explain why the value V_t of a futures contract is 0 at any time $t \in \{0,\dots,T\}$. Show as well that

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=t+1}^{T} d(t,k)\big(G_k(T;S_T) - G_{k-1}(T;S_T)\big)\bigg|\mathcal{F}_t\right].$$

Prove then that $G_T(T;S_T) = S_T$ and deduce from all the above that the futures prices are actually given by

$$G_t(T;S_T) = \mathbb{E}^{\mathbb{Q}}\big[S_T\big|\mathcal{F}_t\big],\ t \in \{0,\dots,T\}.$$

Show then that the difference between forward and futures prices is given by

$$F_t(T;S_T) - G_t(T;S_T) = \frac{\mathbb{C}\mathrm{ov}^{\mathbb{Q}}[S_T, d(t,T)|\mathcal{F}_t]}{B(t,T)},\ t \in \{0,\dots,T\}.$$

Which of $F_t(T;S_T)$ or $G_t(T;S_T)$ would you expect is usually the largest? Why?

By definition, the cash-flows occurring at each payment date are designed to maintain the value of the futures contract to 0. Besides, we know that the value at any time for the futures contract is given by the expectation under any (this is a tradable asset, therefore automatically replicable) risk-neutral measure of the sum of the future discounted cash-flows, hence the second formula. At maturity, it is obvious that the futures price must be equal to the asset price, as this the only price at which the asset can be delivered immediately. Let us now prove the final result by backward induction.

When $t = T$, the result is immediate. Let us assume it is true for some $t \in \{1,\dots,T\}$, and all $\ell \in \{t+1,\dots,T\}$. We then have

$$\begin{aligned}
0 = V_{t-1} &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=t}^{T} d(t-1,k)\big(G_k(T;S_T) - G_{k-1}(T;S_T)\big)\bigg|\mathcal{F}_{t-1}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\big[d(t-1,t)\big(G_t(T;S_T) - G_{t-1}(T;S_T)\big)\big|\mathcal{F}_{t-1}\big] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[d(t-1,t)\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=t+1}^{T} d(t+1,k)\big(G_k(T;S_T) - G_{k-1}(T;S_T)\big)\bigg|\mathcal{F}_t\right]\bigg|\mathcal{F}_{t-1}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\big[d(t-1,t)S_T\big|\mathcal{F}_{t-1}\big] - G_{t-1}(T;S_T)\mathbb{E}^{\mathbb{Q}}\big[d(t-1,t)\big|\mathcal{F}_{t-1}\big] + \mathbb{E}^{\mathbb{Q}}\big[d(t-1,t)V_t|\mathcal{F}_{t-1}\big] \\
&= d(t-1,t)\big(\mathbb{E}^{\mathbb{Q}}\big[S_T\big|\mathcal{F}_{t-1}\big] - G_{t-1}(T;S_T)\big),
\end{aligned}$$

where we used the fact that in discrete–time models, $d(t-1,t)$ is already known at time $t-1$. This obviously gives us the desired result.

Next, we have on the one hand

$$F_t(T;S_T) - G_t(T;S_T) = \frac{S_t}{B(t,T)} - \mathbb{E}^{\mathbb{Q}}\big[S_T\big|\mathcal{F}_t\big],$$

and on the other hand

$$\frac{\mathbb{C}\mathrm{ov}^{\mathbb{Q}}[S_T, d(t,T)|\mathcal{F}_t]}{B(t,T)} = \frac{\mathbb{E}^{\mathbb{Q}}[d(t,T)S_T|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}[d(t,T)|\mathcal{F}_t]\mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]}{B(t,T)} = \frac{S_t}{B(t,T)} - \mathbb{E}^{\mathbb{Q}}\big[S_T\big|\mathcal{F}_t\big],$$

hence the last result.

From this formula we recover first the result seen in the Lecture Notes that when the discount factors are deterministic, forward and futures prices are equal. In addition, we have obtained that forward prices are higher that futures prices when the underling asset is positively correlated with the discount factor. As the discount factor is a decreasing function of the interest rate, we deduce that whenever the value of the asset is negatively correlated with the interest rate, the forward price is higher than the futures price. The answer depends then a bit on the nature of the asset. It is typical that stocks are negatively correlated with bonds, since bonds affect the stock market by competing with stocks for investors' cash. Since bonds are also negatively correlated with interest rates, stocks will tend to be positively correlated with interest rates. This means that forward prices of stocks will tend to be below the corresponding futures prices, while the opposite is true for forward and futures prices of bonds.

Numéraire change and applications

We fix a general complete and arbitrage-free financial market in discrete-time with horizon $T \in \mathbb{N}\setminus\{0\}$, as described in the lecture notes. We let $\mathbb{Q}$ be the unique risk-neutral on this market, and to avoid any issues with integrability requirements, we assume that all assets appearing have bounded prices.

1) Instead of using the risk-less asset S^0 as numéraire, the goal of this question is to examine what happens if we use another asset. Without loss of generality, we will thus the first risky asset S^1. Let ξ be the payoff at time T of an option, and let $(p_t(\xi))_{t\in\{0,\dots,T\}}$ be the associated no-arbitrage price.

Define the probability measure $\mathbb{P}^1$ on $(\Omega,\mathcal{F})$ by

$$\frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{Q}} := \frac{S^1_T}{S^1_0 S^0_T}.$$

1)a) Show that $\mathbb{P}^1$ is well-defined.

Positivity and integrability are obvious. Besides, since $\mathbb{Q}$ is a risk-neutral measure, $\widetilde{S}^1$ is an $(\mathbb{F},\mathbb{Q})$-martingale and

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S^1_T}{S^1_0 S^0_T}\right] = \frac{1}{S^1_0}\mathbb{E}^{\mathbb{Q}}\big[\widetilde{S}^1_T\big] = 1,$$

proving that $\mathbb{P}^1$ is indeed a well-defined probability measure.

1)b) We define the S^1-discounted value of any process $(V_t)_{t\in\{0,\dots,T\}}$ by

$$V^{S^1}_t := \frac{V_t}{S^1_t},\ t\in\{0,\dots,T\}.$$

Prove that $\widetilde{V}$ is an $(\mathbb{F},\mathbb{Q})$-martingale if and only if V^{S^1} is an $(\mathbb{F},\mathbb{P}^1)$-martingale.

Assume that $\widetilde{V}$ is an $(\mathbb{F},\mathbb{Q})$-martingale (and recall that since we assumed all processes bounded, there is no question about integrability here), then by Bayes's formula, we have for any $t\in\{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{P}^1}\left[V^{S^1}_{t+1}\middle|\mathcal{F}_t\right]=\frac{\mathbb{E}^{\mathbb{Q}}\left[V^{S^1}_{t+1}\frac{S^1_T}{S^1_0S^0_T}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{S^1_T}{S^1_0S^0_T}\middle|\mathcal{F}_t\right]}=\frac{\mathbb{E}^{\mathbb{Q}}\left[V^{S^1}_{t+1}\widetilde{S}^1_{t+1}\middle|\mathcal{F}_t\right]}{\widetilde{S}^1_t}=\frac{\mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_{t+1}\middle|\mathcal{F}_t\right]}{\widetilde{S}^1_t}=V^{S^1}_t,$$

which is the desired result. Conversely, if V^{S^1} is an $(\mathbb{F},\mathbb{P}^1)$-martingale, we have for any $t\in\{0,\dots,T-1\}$

$$\mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_{t+1}\middle|\mathcal{F}_t\right]=\frac{\mathbb{E}^{\mathbb{P}^1}\left[\widetilde{V}_{t+1}\frac{S^0_T}{S^1_T}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}^1}\left[\frac{S^0_T}{S^1_T}\middle|\mathcal{F}_t\right]}=\frac{\mathbb{E}^{\mathbb{P}^1}\left[\widetilde{V}_{t+1}(S^0)^{S^1}_{t+1}\middle|\mathcal{F}_t\right]}{(S^0)^{S^1}_t}=\frac{\mathbb{E}^{\mathbb{P}^1}\left[V^{S^1}_{t+1}\middle|\mathcal{F}_t\right]}{(S^0)^{S^1}_t}=\widetilde{V}_t,$$

where we used the fact that since $\widetilde{S}^0$ is an $(\mathbb{F},\mathbb{Q})$-martingale, we know by the first part of the proof that $(S^0)^{S^1}$ is an $(\mathbb{F},\mathbb{P}^1)$-martingale.

1)c) Deduce that we can write

$$p_t(\xi)=\mathbb{E}^{\mathbb{P}^1}\left[\frac{S^1_t}{S^1_T}\xi\middle|\mathcal{F}_t\right].$$

This formula shows that the risk-neutral pricing method is invariant under the so-called *change of numéraire*.

We have

$$p_t(\xi)=\mathbb{E}^{\mathbb{Q}}\left[\frac{S^0_t}{S^0_T}\xi\middle|\mathcal{F}_t\right]=\mathbb{E}^{\mathbb{Q}}\left[\frac{S^1_T}{S^1_0S^0_T}\middle|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{P}^1}\left[\frac{S^1_0S^0_T}{S^1_T}\frac{S^0_t}{S^0_T}\xi\middle|\mathcal{F}_t\right]=S^1_t\mathbb{E}^{\mathbb{P}^1}\left[\frac{\xi}{S^1_T}\middle|\mathcal{F}_t\right],$$

which is the desired result.

2) Caps and floors are to PFS and RFS what call and put options are to forward contracts. In other words, they correspond to IRS contracts where exchange at each payment date only occurs if the payoff is positive. More precisely, we fix a number of payments $n\in\mathbb{N}\setminus\{0\}$, and a sequence $\mathcal{T}:=(T_i)_{i\in\{0,\dots,n\}}$ (all belonging to $\{0,\dots,T\}$) of dates, as well as a face–value $N>0$ and a strike $K>0$. The discounted payoff at time $t\leq T_0$ of a cap is given by

$$N\sum_{i=1}^n d(t,T_i)(T_i-T_{i-1})\big(\ell(T_{i-1},T_i)-K\big)^+,$$

while that of a floor is

$$N\sum_{i=1}^n d(t,T_i)(T_i-T_{i-1})\big(K-\ell(T_{i-1},T_i)\big)^+.$$

Caps and floors are actually constituted of a stream of simpler contracts, with discounted payoffs of the form

$$Nd(t,T_i)(T_i-T_{i-1})\big(\ell(T_{i-1},T_i)-K\big)^+,\text{ and }Nd(t,T_i)(T_i-T_{i-1})\big(K-\ell(T_{i-1},T_i)\big)^+,$$

which are called respectively the i-th caplet associated to the cap, and the i-th floorlet associated to the floor.

2)a) For any $t\in\{0,\dots,T_0\}$, we let $\mathrm{CAPL}_t(T_{i-1},T_i,N,K)$ be the price at t of the i-th caplet associated to the cap, and $\mathrm{FLOORL}_t(T_{i-1},T_i,N,K)$ the price at t of the i-th floorlet associated to the floor. Show that

$$\mathrm{CAPL}_t(T_{i-1},T_i,N,K)=N(1+(T_i-T_{i-1})K)\mathrm{ZBP}_t\big(T_{i-1},T_i,(1+(T_i-T_{i-1})K)^{-1}\big),$$
$$\mathrm{FLOORL}_t(T_{i-1},T_i,N,K)=N(1+(T_i-T_{i-1})K)\mathrm{ZBC}_t(T_{i-1},T_i,(1+(T_i-T_{i-1})K)^{-1}\big),$$

where for any $0\leq t\leq k\leq s\leq T$, $\mathrm{ZBC}_t(k,s,L)$ is the value at t of a call option with maturity k, written on a zero-coupon bond with maturity s, and with strike $L\geq 0$, and $\mathrm{ZBP}_t(k,s,L)$ is the value at t of a put option with maturity k, written on a zero-coupon bond with maturity s, and with strike $L\geq 0$.

We only give the result for the caplets, the one for the floorlets being immediate by call–put parity. Notice that by conditioning with respect to $\mathcal{F}_{T_{i-1}}$, we have directly

$$\begin{aligned}\mathrm{CAPL}_t(T_{i-1},T_i,N,K) &= N\mathbb{E}^{\mathbb{Q}}\big[d(t,T_i)(T_i-T_{i-1})\big(\ell(T_{i-1},T_i)-K\big)^+\big|\mathcal{F}_t\big]\\ &= N\mathbb{E}^{\mathbb{Q}}\Big[d(t,T_{i-1})(T_i-T_{i-1})\big(\ell(T_{i-1},T_i)-K\big)^+\mathbb{E}^{\mathbb{P}^\lambda}\big[d(T_{i-1},T_i)\big|\mathcal{F}^{W,\mathbb{P}}_{T_{i-1}}\big]\Big|\mathcal{F}_t\Big]\\ &= N\mathbb{E}^{\mathbb{Q}}\big[d(t,T_{i-1})B(T_{i-1},T_i)(T_i-T_{i-1})\big(\ell(T_{i-1},T_i)-K\big)^+\big|\mathcal{F}_t\big].\end{aligned}$$

Using next the definition of the simply-compounded forward rate, we deduce

$$\begin{aligned}\mathrm{CAPL}_t(T_{i-1},T_i,N,K) &= N\mathbb{E}^{\mathbb{Q}}\left[d(t,T_{i-1})B(T_{i-1},T_i)\left(\frac{1}{B(T_{i-1},T_i)}-\big(1+(T_i-T_{i-1})K\big)\right)^+\middle|\mathcal{F}_t\right]\\ &= N(1+(T_i-T_{i-1})K)\mathbb{E}^{\mathbb{Q}}\left[d(t,T_{i-1})\left(\frac{1}{1+(T_i-T_{i-1})K}-B(T_{i-1},T_i)\right)^+\middle|\mathcal{F}_t\right],\end{aligned}$$

from which the result is immediate.

2)b) Now for any $s\in\{0,\dots,T\}$, we let the s-forward martingale measure (or forward martingale measure with maturity s) correspond to choosing the zero-coupon bond with maturity s as numéraire. This of course means that the measure $\mathbb{P}^s:=\mathbb{P}^{B(\cdot,s)}$ is defined on the probability space $(\Omega,\mathcal{F}_s)$ only, since $B(\cdot,s)$ ceases to exist after time s. Show that for any $0\le t\le k\le s\le T$

$$\begin{aligned}\mathrm{ZBC}_t(k,s,K) &= B(t,k)\mathbb{E}^{\mathbb{P}^k}\big[\big(F_k(k;B(k,s))-K\big)^+\big|\mathcal{F}_t\big],\\ \mathrm{ZBP}_t(k,s,K) &= B(t,k)\mathbb{E}^{\mathbb{P}^k}\big[\big(K-F_k(k;B(k,s))\big)^+\big|\mathcal{F}_t\big].\end{aligned}$$

We use 1)c) **from which we get**

$$\mathrm{ZBC}_t(k,s,K)=\mathbb{E}^{\mathbb{P}^k}\left[\frac{B(t,k)}{B(k,k)}\big(B(k,s)-K\big)^+\middle|\mathcal{F}_t\right].$$

It then suffices to notice that $B(k,k)=1$, and that since a zero-coupon bond has zero storage costs, we have

$$F_k(k;B(k,s))=\frac{B(k,s)}{B(k,k)}.$$

The exact same proof works for the put case.

2)c) Deduce a formula for the cap and the floor described above in terms of expectations under the forward martingale measures with maturities $(T_i)_{i\in\{0,\dots,n-1\}}$.

We only give the formula in the cap case. We have

$$\begin{aligned}\mathrm{CAP}_t(\mathcal{T},N,K) &= \sum_{i=1}^n \mathrm{CAPL}_t(T_{i-1},T_i,N,K)\\ &= N\sum_{i=1}^n(1+(T_i-T_{i-1})K)\mathrm{ZBP}_t\big(T_{i-1},T_i,(1+(T_i-T_{i-1})K)^{-1}\big)\\ &= N\sum_{i=1}^n(1+(T_i-T_{i-1})K)B(t,T_{i-1})\mathbb{E}^{\mathbb{P}^{T_{i-1}}}\Big[\big(K-F_{T_{i-1}}\big(T_{i-1};B(T_{i-1},T_i)\big)\big)^+\Big|\mathcal{F}_t\Big].\end{aligned}$$

2)d) Which advantage do you see in these formulae compared to the ones written under the risk-neutral measure $\mathbb{Q}$?

The main point here is that when pricing under $\mathbb{Q}$, we need to compute expectations of quantities involving both the discount factors d and the zero-coupon bond prices. When rates are not deterministic, this is typically a complicated task. However, when using the forward martingale measures,

though we have now n expectations to compute instead of 1, we managed to get rid off the discount factors. Of course, the question is then whether one can 'easily' say something about the distribution of the forward prices of the bonds under the forward measure. It turns out that in a large class of model for the term-structure of interest rates, this is actually doable. The above trick then becomes very useful.

3) We now consider *swaptions*. These are simply options whose underlying is an IRS. As usual, there are two main types of such options: the payer one (call-like), and the receiver one (put-like). More precisely, a European payer swaption is an option giving the right (and thus not the obligation) to enter a payer IRS at a given future time, which is called the swaption maturity. Usually, this maturity coincides with the first reset date of the underlying IRS. The underlying IRS length, that is to say $T_n - T_0$ with our previous notations, is called the *tenor* of the swaption. It is also commonplace to call *tenor structure* the set of reset and payment dates $\mathcal{T}$. Using T_0 as our maturity date, the payoff at maturity of a payer-swaption, discounted from some time $0 \leq t \leq T_0$ is therefore given by

$$d(t,T_0)\big(\mathrm{PFS}_{T_0}(\mathcal{T},N,K)\big)^+.$$

3)a) Show that

$$\mathrm{PFS}_{T_0}(\mathcal{T},N,K) = N\sum_{i=1}^n B(T_0,T_i)(T_i - T_{i-1})\big(\ell(T_0,T_{i-1},T_i) - K\big).$$

See the Lecture Notes, Example 1.4.4.

3)b) Show that the payoff of a payer-swaption can then be rewritten

$$Nd(t,T_0)\big(s(T_0,\mathcal{T}) - K\big)^+\sum_{i=1}^n (T_i - T_{i-1})B(T_0,T_i),$$

where the forward–swap rate $s(t,\mathcal{T})$ at time t for the sets of times $\mathcal{T}$ is given by

$$s(t,\mathcal{T}) := \frac{B(t,T_0) - B(t,T_n)}{\sum_{i=1}^n (T_i - T_{i-1})B(t,T_i)}.$$

One simply needs to remember that the simply-compounded forward rates satsify

$$\ell(T_0,T_{i-1},T_i) = \frac{1}{T_i - T_{i-1}}\left(\frac{B(T_0,T_{i-1})}{B(T_0,T_i)} - 1\right).$$

Then

$$\begin{aligned}
d(t,T_0)\big(\mathrm{PFS}_{T_0}(\mathcal{T},N,K)\big)^+ &= d(t,T_0)\left(N\sum_{i=1}^n B(T_0,T_i)(T_i - T_{i-1})\big(\ell(T_0,T_{i-1},T_i) - K\big)\right)^+ \\
&= d(t,T_0)\left(N\sum_{i=1}^n \big(B(T_0,T_{i-1}) - B(T_0,T_i)\big) - NK\sum_{i=1}^n (T_i - T_{i-1})B(T_0,T_i)\right)^+ \\
&= d(t,T_0)\left(N\big(B(T_0,T_0) - B(T_0,T_n)\big) - NK\sum_{i=1}^n (T_i - T_{i-1})B(T_0,T_i)\right)^+ \\
&= Nd(t,T_0)\big(s(T_0,\mathcal{T}) - K\big)^+\sum_{i=1}^n (T_i - T_{i-1})B(T_0,T_i).
\end{aligned}$$

3)*c*) Let us define the so-called *level process* G by

$$G_t := \sum_{i=1}^{n}(T_i - T_{i-1})B(t, T_i).$$

We let Π correspond to the probability measure defined in 1) when choosing G as a numéraire. Show that the forward–swap rate $s(\cdot, \mathcal{T})$ is an $(\mathbb{F}, \Pi)$-martingale, and then that if $\mathrm{PSWAP}_t(\mathcal{T}, N, K)$ represents the value at time $0 \leq t \leq T_0$ of the payer-swaption, we have

$$\mathrm{PSWAP}_t(\mathcal{T}, N, K) = NG_t\mathbb{E}^{\Pi}\big[(s(T_0, \mathcal{T}) - K)^+ \big| \mathcal{F}_t\big].$$

Since the zero-coupon bonds with maturity T_0 and T_n are risky assets, then their discounted values must be $(\mathbb{F}, \mathbb{Q})$-martingales. By $1)b)$, this implies that $\frac{B(\cdot, T_0)}{G_\cdot}$ and $\frac{B(\cdot, T_n)}{G_\cdot}$ are $(\mathbb{F}, \Pi)$-martingales, and thus so is their difference. This shows that the forward–swap rate is an $(\mathbb{F}, \Pi)$-martingale. Then, we have by $1)c)$

$$\begin{aligned}\mathrm{PSWAP}_t(\mathcal{T}, N, K) = \mathbb{E}^{\mathbb{Q}}\Big[Nd(t, T_0)\big(s(T_0, \mathcal{T}) - K\big)^+ G_{T_0}\Big|\mathcal{F}_t\Big] &= \mathbb{E}^{\Pi}\left[\frac{G_t}{G_{T_0}}N\big(s(T_0, \mathcal{T}) - K\big)^+ G_{T_0}\middle|\mathcal{F}_t\right] \\ &= NG_t\mathbb{E}^{\Pi}\big[(s(T_0, \mathcal{T}) - K)^+\big|\mathcal{F}_t\big].\end{aligned}$$