Spectral theory of hyperbolic surfaces

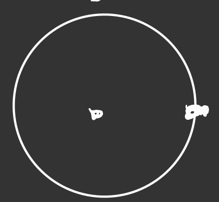
Today: Elements of non-Euclidean and Riemannian geometry

§. Non-Euclidean geometry

5 postulates of Euclidean geometry

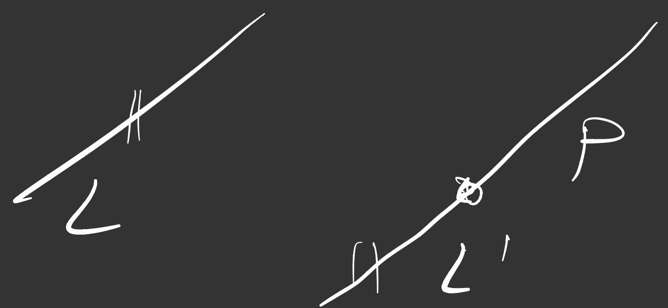
1. 

2. 

3. 

4. 

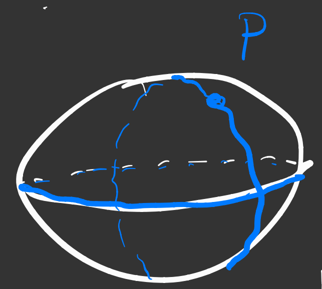
5. Given a line L and a pt. $P \notin L$, $\exists!$ L' that passes through P and is parallel to L



Lobachevski, Bolyai, Gauss:

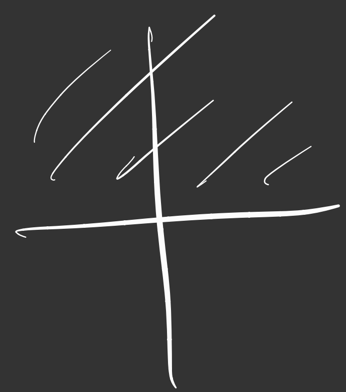
You can have consistent geom. with only 1–4.

Two models in which the parallel postulate (5.) fails:

1. 

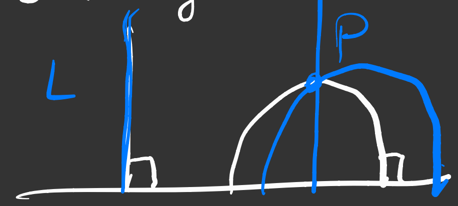
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$$

line = great circle

2. 

$$\mathbb{H}^2 = \{z \in \mathbb{C} : y > 0\}$$

$z = x+iy$

line 

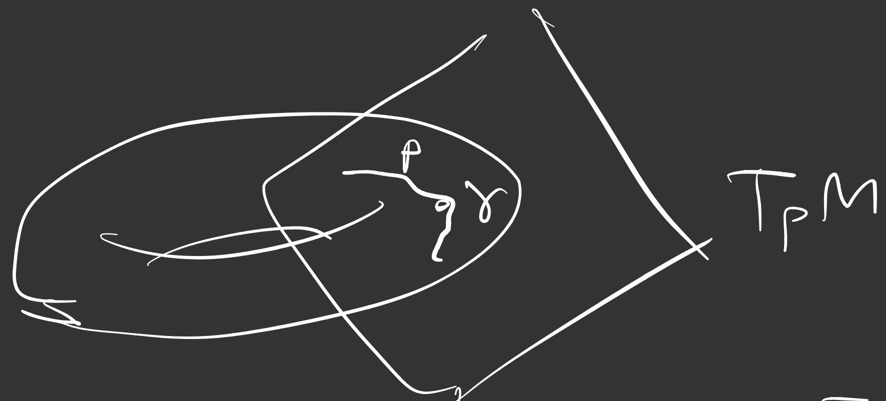
§. Geometry of curved surfaces

<u>def</u>: A smooth surface in $\mathbb{R}^3$ is a connected Hausdorff topo. space where each pt. admits a nbhd. that is diffeomorphic to an open subset of $\mathbb{R}^2$.

$$F = f^{-1} : U \subset \mathbb{R}^2 \longrightarrow M$$

is called a local parametr. near p.

<u>Fact</u>: An inj C^∞ $F : U \subset \mathbb{R}^2 \to M$ is a local parametrization iff $\partial_x F$ and $\partial_y F$ are linearly independent at each pt of U.



$$T_p M = \operatorname{span}\{\partial_x F, \partial_y F\} \cong \mathbb{R}^2$$

$\gamma : [a,b] \longrightarrow M$ C^∞ curve has length

$$L(\gamma) = \int_a^b \| \gamma'(t) \| \, dt$$

Here $\|\cdot\|$ is the std. Eucl. norm on $\mathbb{R}^3$.

<u>Ex</u>:

- L does not depend on the particular parametrization of the curve.

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$$

Locally,

$$\gamma(t) = F(\gamma_{loc}(t))$$

$$\gamma_{loc}(t) = (x(t), y(t)) \in \mathbb{R}^2$$

$$\gamma'(t) = DF_{\gamma_{loc}(t)}\, \gamma_{loc}'(t)$$

$$\|\gamma'(t)\|^2 = \gamma'(t) \cdot \gamma'(t)$$

$$= \gamma_{loc}'(t)^T\, DF^T DF\, \gamma_{loc}'(t)$$

$$\gamma_{loc}'(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

$$= (dx\ dy)\, DF^T DF \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$DF^T DF = \begin{pmatrix} \|\partial_x F\|^2 & \partial_x F \partial_y F \\ \partial_y F \partial_x F & \|\partial_y F\|^2 \end{pmatrix}$$

$$\hookrightarrow (\partial_x F \mid \partial_y F)$$

$$= \|\partial_x F\|^2 dx^2 + 2(\partial_x F \cdot \partial_y F)\, dx\, dy + \|\partial_y F\|^2 dy^2 = ds^2$$

(FFF) <u>first fundamental form</u>

$ds = \sqrt{ds^2}$ line element

$$L(\gamma) = \int_\gamma ds = \int_a^b \|\gamma'(t)\|\, dt$$

<u>Ex</u>: Check that this is invariant under change of local coordinates.

<u>An element of the importance of the FFF</u>

def: Let M_1, M_2 be two C^∞ surfaces in $\mathbb{R}^3$. A diffeomorphism $\phi: M_1 \to M_2$ is an *isometry* if it preserves the lengths of curves, i.e. $\forall$ C^∞ curve $\gamma: [a,b] \to M_1$

$$L(\phi \circ \gamma) = L(\gamma).$$

Thm: Two C^∞ surfaces M_1, M_2 in $\mathbb{R}^3$ are locally isometric near $p_1 \in M_1$, $p_2 \in M_2$ iff they admit local parametrization F_1 and F_2 near p_1 and p_2 with the same first fundamental form.

Proof:

$\Leftarrow$ is immediate from the local computation of length

$\Rightarrow$ $\phi: M_1 \to M_2$ loc. isometry

F_1 local param. near p_1

$F_2 = \phi \circ F_1$ is a local param. near p_2

$$L(F_2 \circ \gamma_{loc}) = L(\phi \circ F_1 \circ \gamma_{loc})$$

$$= L(F_1 \circ \gamma_{loc})$$

claim: Length determines the first fundamental form locally.

Set

$$\gamma_{loc}(t) = p + \begin{pmatrix} t \\ 0 \end{pmatrix} \qquad \gamma_{loc}'(t) = e_1$$

$$\gamma = F(\gamma_{loc})$$

$$\frac{d}{d\varepsilon} L(\gamma|_{[0,\varepsilon]}) = \frac{d}{d\varepsilon}\int_0^{\varepsilon} \|\partial_x F\| \, dt$$

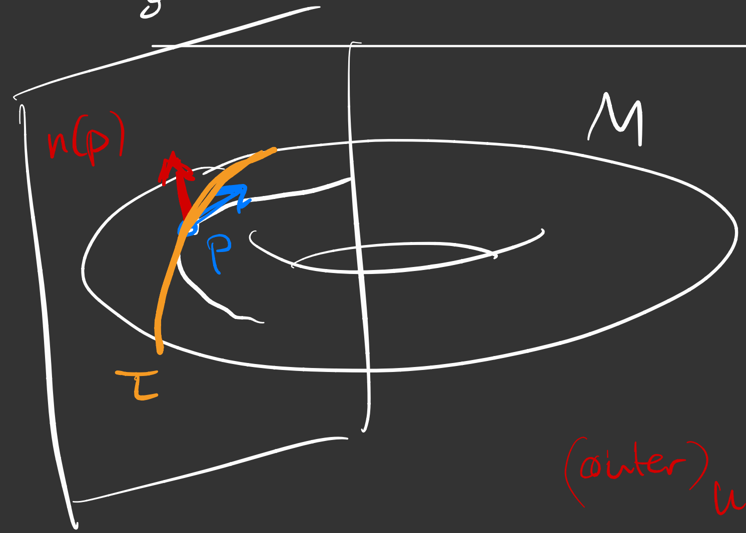
$$= \left\| \partial_x F\big|_{p+\binom{\varepsilon}{0}} \right\|$$

Now take $\varepsilon = 0$.

If you start with $\gamma_{loc}(t) = p + \begin{pmatrix} 0 \\ t \end{pmatrix}$, you can determine $\|\partial_y F|_p\|$

If you go with $\gamma_{loc}(t) = p + \begin{pmatrix} t \\ t \end{pmatrix}$ ... $\partial_x F \cdot \partial_y F$

§. Gaussian curvature



$\gamma'(t)$

$p = \gamma(t)$

$n(p)$ is the (outer) unit normal vector to T_pM

Π is the plane determined by $n(p)$ and $\gamma'(t)$

$\tau = \Pi \cap M$ is a plane curve in Π with curvature K_Π

Doing the same for all $v \in T_pM$, the <u>principle curvatures</u> K_1 and K_2 are the max. and min. curvatures obtained.

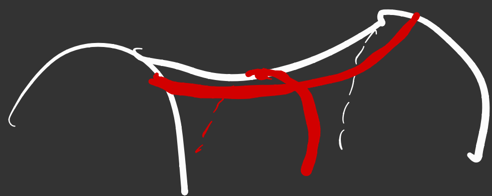
The corresponding vectors are called the *principal directions*

def: The *Gaussian curvature* at p is $K(p) = \kappa_1(p) \cdot \kappa_2(p)$ the product of the principal curvatures at p.

$K = 0$

$K > 0$

$K < 0$



Theorema Egregium

On a C^∞ surface in $\mathbb{R}^3$, Gaussian curvature is completely determined by the first fundamental form.

This means, Gaussian curvature is invariant under local isometries.

§. Riemannian geometry (or elements thereof)

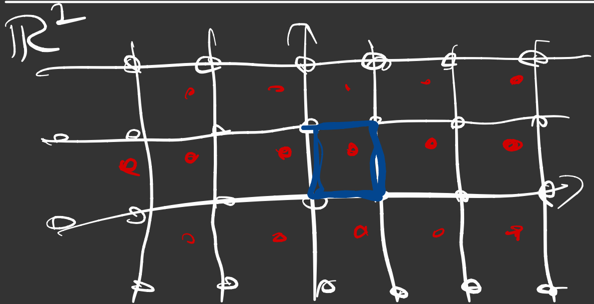
Riem. geometry builds on

1. The def. of (abstract) C^∞ manifolds

2. Riemannian metric

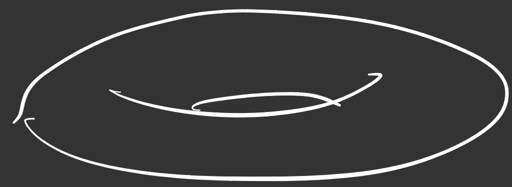
Riemann's proposition: The Riemannian metric is all you need to study the intrinsic geometry of a C^∞ mfld.

3. Non-Euclidean geometries

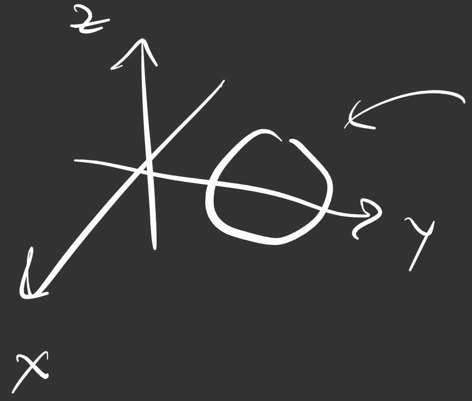


$\mathbb{Z}^2$

1.

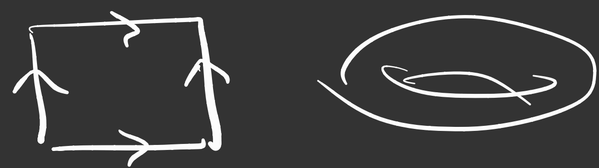


can be realized as a surface of revolution



circle in y-z-plane and rotate around z-axis

or we can realize the torus as



$$\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$$

$$(x, y) \sim (x+m, y+n)$$

$$\forall m, n \in \mathbb{Z}$$

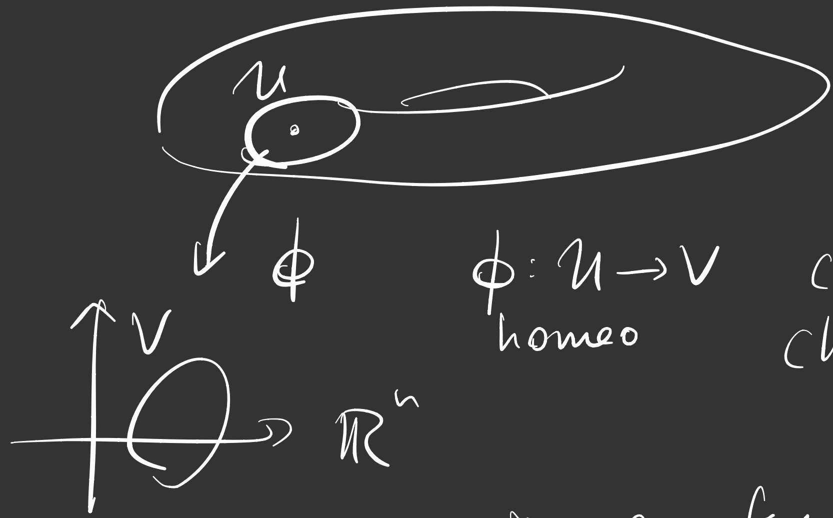
$f: \mathbb{T}^2 \longrightarrow \mathbb{R}$ $\quad C^\infty$

correspond to $\tilde{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad C^\infty$

s.t. $\quad f(x+\xi) = f(x) \quad \forall \xi \in \mathbb{Z}^2$

<ins>def</ins>: A <ins>n-dimensional manifold</ins> M is a connected, second-countable, Hausdorff topological space s.t. each pt. has a nbhd that is homeom. to $\mathbb{R}^n$.

A <ins>smooth n-dim. mfld</ins> is such a mfld that admits a maximal smooth atlas.



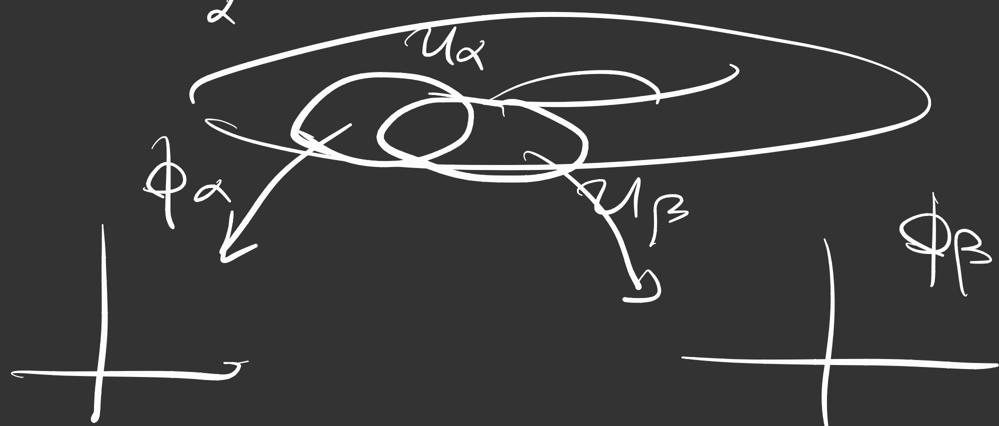
$\phi: U \to V$ coordinate chart

homeo

An atlas is a family

$\{(U_\alpha, \phi_\alpha)\}_\alpha$ s.t.

$$\bigcup_\alpha U_\alpha = M$$



$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$

we say that $\{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas if these "transition maps" are diffeomorphisms.

2) Let M be a C^∞ mfld.

def: A Riemannian metric on M is a family $(g_p)_{p\in M}$ of inner products,

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

bilinear, symmetric, pos. definite, that vary differentiably in p.

g_p define
- $\|v\|_p = \sqrt{g_p(v,v)}$
- angles btw. vectors in T_pM
- $ds^2 = g_{ij}\, dx_i\, dx_j$
- notions of curvature ...

- γ a C^∞ curve on M

then
$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

def: A Riemannian mfld. is a C^∞ mfld equipped with a Riemannian metric.

Fact: Every C^∞ mfld. admits a Riemannian metric.

Exples:

- $\mathbb{R}^2$ $\quad g_p(u,v) = u \cdot v$ $\quad K=0$

$$ds^2 = dx^2 + dy^2$$

- $\mathbb{S}^2 = \{(\cos x \sin y, \sin x \sin y, \cos y) : x, y \in [0, 2\pi]\}$

$K=1$

$$ds^2 = \sin^2 y \, dx^2 + dy^2$$

- $\mathbb{H}^2$ with the hyperbolic metric

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

$K=-1$

Fact: A C^∞ surface with constant curvature

$K=0$ then it is locally isometric to $\mathbb{R}^2$

$K=1$ ... $\mathbb{S}^2$

$K=-1$ ... $\mathbb{H}^2$

def: A hyperbolic surface is a C^∞ surface equipped with a Riemannian metric of constant curvature $=-1$.