Last time: Deducing solution of heat equation via the Selberg/spherical transform.

... $(p_t(z,w))_{t>0}$ family of point-pair invariants s.t.

$$u(z,t) = \int_{\mathbb{H}} p_t(z,w)\, f(w)\, d\mu(w)$$

verifies

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta u \\[2ex] \lim\limits_{t\to 0} u(z,t) = f(z) \end{cases}$$

Fix Γ Fuchsian s.t. $M = \Gamma\backslash\mathbb{H}$ compact. and build the corresponding automorphic kernel

$$P_t(z,w) = \sum_{\gamma\in\Gamma} p_t(z,\gamma w)$$

We left at: $\exists C_0 > 0$

$$|P_t(z,w)| \le C_0 \frac{1}{t} \sum_{n\geq 0} \#\{z'\in\Gamma w : n \le d(z,z') < n+1\}\, e^{-n^2/8t}$$

Still need to find an upper bound for

$$N_T = \#\{z'\in\Gamma w : d(z,z') < T\}$$

(when T is large.)

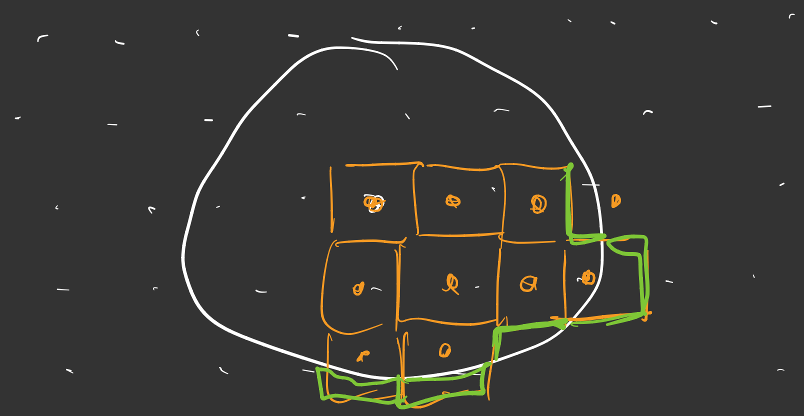
Today:

- Euclidean & hyperbolic circle problem
- Proof of the spectral thm. for compact hyperbolic surfaces via heat kernels.
- Application to the hyperbolic circle problem

Next time: Selberg's trace formula

Euclidean circle problem

$$N_T := \#\{\xi \in \mathbb{Z}^2 : \|\xi\| < T\}$$



Circle problem consists in estimating N_T precisely for large T's.

$$N_T = \#\left\{\text{unit } \square \text{ centered at } \xi \in \mathbb{Z}^2 \cap D_T\right\}$$

$$\left| \text{area}\left(\bigcup\{\square \text{ with center } \xi \in \mathbb{Z}^2 \cap D_T\}\right) - \text{area } D_T \right| \leq$$

$$\pi\left(T + \frac{\sqrt{2}}{2}\right)^2 - \pi\left(T - \frac{\sqrt{2}}{2}\right)^2$$

$$= 2\sqrt{2}\pi T$$

$$N_T = \pi T^2 + O(T)$$

Gauss circle problem (Conjecture)

$$N_T = \pi T^2 + O(T^{1/2+\varepsilon}) \quad \forall \varepsilon > 0$$

(Hardy, 1917)

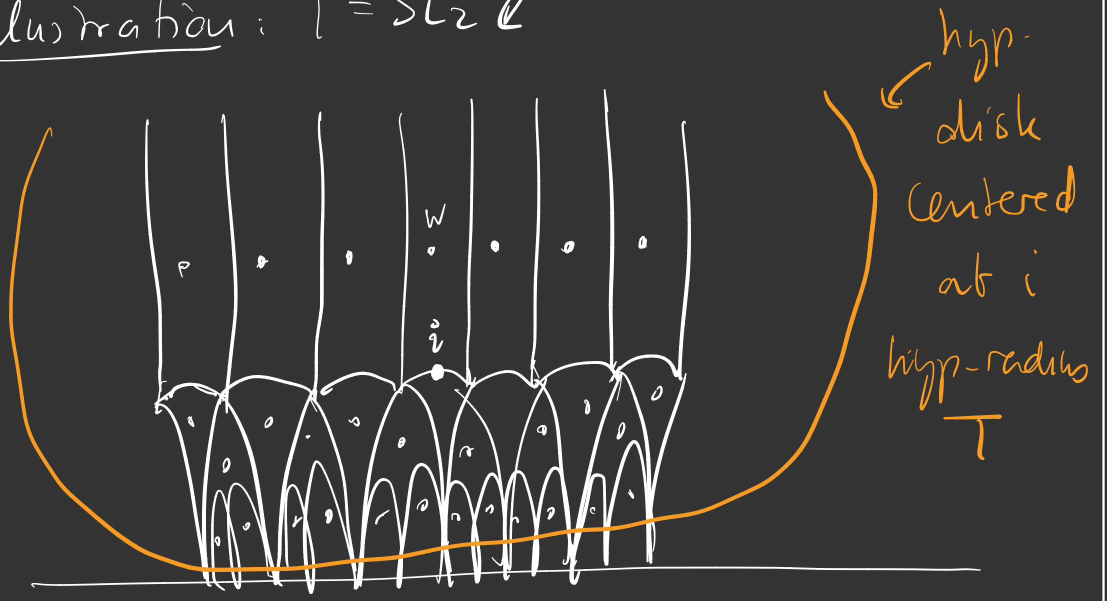
Sierpinski (1906) $O(T^{2/3})$

⋮

Bourgain, Watt (2017) $O(T^{0.63\ldots})$

Hyperbolic circle problem

Illustration: $\Gamma = SL_2\mathbb{Z}$



$$N_T = \#\{ z' \in \Gamma w : d(z', z) < T \}$$

Now choose Γ without parabolics (i.e. $\mathcal{F}$ has no vertex on $\partial \mathbb{H}$)

$$= \frac{\operatorname{area}\left(\bigcup \mathcal{F} \text{ containing } z' \in \Gamma w \text{ with } d(z, z') < T \right)}{\operatorname{area} \mathcal{F}}$$

$$\leq \frac{\operatorname{area}\left(D_{T + \operatorname{diam} \mathcal{F}} \right)}{\operatorname{area} \mathcal{F}}$$

Recall: If D_T is a hyperbolic disk, then

$$\operatorname{area} D_T = 4\pi \sinh\left(\frac{T}{2}\right)^2$$

$$\approx \pi e^T$$

(for T very large)

Coming back to our autom. kernel

$$|P_t(z, w)| \leq C \cdot \frac{1}{t} \sum_{n \geq 0} e^{n - n^2/8t}$$

$$< \infty$$

(exercise)

Prop: M cpt. hyp. surface. Then

$$u(z,t) = \int_M P_t(z,w) f(w)\, d\mu(w)$$

is the unique solution to

$$(*) \begin{cases} \dfrac{\partial}{\partial t} u + \Delta u = 0 \\ \lim\limits_{t \to 0} u(z,t) = f(z) \end{cases}$$

Proof: Suppose u_1 and u_2 ~~are~~ two solutions to $(*)$.

We show that $v = u_1 - u_2 = 0$.

By assumption

$$\begin{cases} \dfrac{\partial}{\partial t} v + \Delta v = 0 \\ \lim\limits_{t \to 0} v(z,t) = 0 \qquad (**) \end{cases}$$

$$\frac{\partial}{\partial t} \|v\|_2^2 = -\int_M \Delta v \cdot v$$

$$= -\int_M (\nabla v)^2 \leq 0$$

Together with $(**)$, we conclude that $v \equiv 0$. $\square$

Thm: M cpt. hyperbolic surface. Then $\exists$ a complete ONB $\{\varphi_k\}_{k \geq 0}$ in $L^2(M)$ s.t.

$$\Delta \varphi_k = \lambda_k \varphi_k \quad \text{and}$$

$$0 = \lambda_0 < \lambda_1 \leq \cdots\cdots \longrightarrow \infty$$

<u>lemma 1</u>: For each $t > 0$,

$$\mathcal{P}_t : L^2(M) \longrightarrow L^2(M)$$

$$\mathcal{P}_t f(z) = \int_M P_t(z,w) f(w)\, d\mu(w)$$

verifies

1. $\mathcal{P}_t$ is continuous in t
2. $\lim_{t \to 0} \mathcal{P}_t = 1$
3. $\mathcal{P}_t$ compact and selfadjoint
4. $\mathcal{P}_s \circ \mathcal{P}_t = \mathcal{P}_{s+t} = \mathcal{P}_t \circ \mathcal{P}_s$

<u>pf</u>: exercise.

<u>lemma 2</u>:

$\exists$ an ONB $\{\varphi_j\}_{j \geq 0}$ in $L^2(M)$ that are simultaneous eigenfunctions of all $\mathcal{P}_t$

and $\quad \mathcal{P}_t \varphi_k = \eta_k^t \varphi_k$

(where $\mathcal{P}_1 \varphi_k = \eta_k \varphi_k$)

<u>Proof</u>:

<u>$t=1$</u>: By the spectral thm, $\exists$ ONB $\{\varphi_k\}_{k \geq 0}$ with $\mathcal{P}_1 \varphi_k = \eta_k \varphi_k$

Suppose $\mathcal{P}_{1/q} \varphi = \mu \varphi$

$$\Rightarrow \mathcal{P}_{1/q}^q \varphi = \mu^q \varphi \overset{4.}{=} \mathcal{P}_1 \varphi$$

$\Rightarrow \varphi$ is an eigenft. of $\mathcal{P}_1$

$\Rightarrow$ complete families of eigenfunctions for $\mathcal{P}_1$ and $\mathcal{P}_{1/q}$ coincide

Lemma 3: For the eigenvalues $\{\eta_k\}_{k\geq 0}$ in lemma 2, we have

$$1 = \eta_0 > \eta_1 \geq \cdots\cdots \longrightarrow 0.$$

Proof:
By lemma 2, $P_t \varphi_k = \eta_k^t \varphi_k$

- Since $\lim_{t\to 0} P_t = 1 \rightsquigarrow \lim_{t\to 0} \eta_k^t = 1$

$\Longrightarrow \eta_k > 0$ for each $k \geq 0$.

- $\langle P_t \varphi_k, \varphi_k \rangle = \eta_k^t \|\varphi_k\|^2$

$$\frac{\partial}{\partial t} \langle \underbrace{P_t \varphi_k}, \varphi_k \rangle = \log \eta_k \cdot \eta_k^t \|\varphi_k\|^2$$

$\hookrightarrow (\Delta + \frac{\partial}{\partial t}) P_t \varphi_k = 0$

$$= -\langle \Delta P_t \varphi_k, \varphi_k \rangle = -\eta_k^t \underbrace{\langle \Delta \varphi_k, \varphi_k \rangle}_{\geq 0}$$

If $\varphi = \varphi_k$, then $\mu^q = \eta_k$

$\Rightarrow P_{1/q} \varphi_k = \eta_k^{1/q} \varphi_k$

$t \in \mathbb{Q}_{\geq 0}$

$$P_{p/q} \varphi_k = P_{1/q}^p \varphi_k = \eta_k^{p/q} \varphi_k$$

Since P_t are continuous in t, this extends to all $t > 0$

$\square$

$$\Rightarrow \log \eta_k = -\frac{\langle \Delta \varphi_k, \varphi_k \rangle}{\|\varphi_k\|^2} \leq 0$$

and $= 0$ iff $\varphi_k =$ constant.

$\Rightarrow \eta_k \in (0, 1]$

$\eta_0 = 1$. Take $\varphi_0 = \frac{1}{\sqrt{\text{area } M}}$

s.t. $\int_M \varphi_0^2 = 1$

and $\mathcal{P}_t \varphi_0(z) = \frac{1}{\sqrt{|M|}} \int_{\mathbb{H}} P_t(z,w)\, d\mu(w)$

is a solution to

$$(*) \begin{cases} \left(\frac{\partial}{\partial t} + \Delta\right) \mathcal{P}_t \varphi_0 = 0 \\ \lim_{t \to 0} \mathcal{P}_t \varphi_0 = \varphi_0 \end{cases}$$

$\tilde{\varphi}_0(z,t) = \varphi_0(z) = \frac{1}{\sqrt{|M|}}$ is also a solution to $(*)$. By uniqueness of the sol. to the heat equation: $\mathcal{P}_t \varphi_0 = \varphi_0$.

- If φ is an eigenft. of $\mathcal{P}_t$ that is not constant,

$$\mathcal{P}_t \varphi = \eta \varphi$$

satisfying $\log \eta = - \frac{\langle \Delta \varphi, \varphi \rangle}{\|\varphi\|^2} < 0$

$\Longrightarrow \eta_k < 1$ for $k \geq 1$.

<u>Thm</u>: M cpt. hyperbolic surface. Then $\exists$ a complete ONB $\{\varphi_k\}_{k \geq 0}$ in $L^2(M)$ s.t.

$\Delta \varphi_k = \lambda_k \varphi_k$ and

$0 = \lambda_0 < \lambda_1 \leq \cdots \longrightarrow \infty$

<u>Proof</u>:

We use that Δ commutes with each $\mathcal{P}_t$ $(t > 0)$ to deduce existence of $\{\varphi_k\}_{k \geq 0}$ with $\Delta \varphi_k = \lambda_k \varphi_k$.

Using that

$$0 = \left(\frac{\partial}{\partial t} + \Delta\right) \mathcal{P}_t \varphi_k = \left(\frac{\partial}{\partial t} + \Delta\right) \eta_k^t \varphi_k$$

$$= (\log \eta_k) \eta_k^t \varphi_k + \eta_k^t \Delta \varphi_k$$

$$\Longrightarrow \Delta \varphi_k = -\log \eta_k \cdot \varphi_k$$

with $1 = \eta_0 \geq \eta_1 \geq \cdots \to 0$

corresponding to

$0 = \lambda_0 < \lambda_1 \leq \cdots \to \infty$ $\square$

Back to hyperbolic circle problem

$N_T = \#\{ z' \in \Gamma w : d(z, z') < T \}$

(Suppose again Γ has only hyperbolic elements.)

$$N_T = \sum_{\gamma \in \Gamma} \mathbb{1}_{[0,T)}(d(z, \gamma w)) = K_T(z,w)$$

$\boxed{\text{Suppose}}$: Fix $w \in \mathbb{H}$ ($\{\varphi_j\}$ ONB for Δ)

$$K_T(z,w) = \sum_{j \geq 0} \langle K_T, \varphi_j \rangle \varphi_j(z) = \sum_{j \geq 0} \hat{\mathbb{1}}_T(\lambda_j) \varphi_j(z) \overline{\varphi_j(w)}$$

$$\langle K_T(\cdot, w), \varphi_j \rangle = \int_{\mathbb{H}} \mathbb{1}_T(z,w) \overline{\varphi_j(z)} \, d\mu(z) = \hat{\mathbb{1}}_T(\lambda_j) \overline{\varphi_j(w)}$$

the Selberg transform

$$\mathbb{1}_T(z,w) = \mathbb{1}_{[0,T)}(d(z,w))$$

$$\lambda_0 = 0 \qquad \hat{\mathbb{1}}_T(0) = \int_{\mathbb{H}} \mathbb{1}_T(i, w)\, \underbrace{\omega_0(w; \mathring{\tau})}_{=1}\, d\mu(w) = \text{area } B_T(i)$$

$$= \text{area of a hyperbolic disc of radius } T$$

$$\varphi_0 = \frac{1}{\sqrt{|M|}}$$

$$N_T = \frac{\text{area}(B_T)}{\text{area}(M)} + \sum_{j \geq 1} \hat{\mathbb{1}}_T(\lambda_j)\, \varphi_j(z)\, \overline{\varphi_j(w)}$$