$\{\varphi_j\}_{j\geq 0}$ ONB in $L^2(M)$

M compact hyperbolic surface

$k \in C_c^\infty(\mathbb{R})$ even

$M = \Gamma \backslash \mathbb{H}$

$$K(z,w) = \sum_{\gamma\in\Gamma} \underbrace{k(z,\gamma w)}_{k(d_{\mathbb{H}}(z,\gamma w))}$$

$$= \sum_{j\geq 0} \hat{k}(\lambda_j)\, \varphi_j(z)\, \overline{\varphi_j(w)}$$

$$\langle K(\cdot,w), \varphi_j\rangle = \int_M K(z,w)\,\overline{\varphi_j(z)}\,d\mu(z)$$

folding/

$$= \int_{\mathbb{H}} k(z,w)\,\overline{\varphi_j(z)}\,d\mu(z)$$

unfolding

$$\Delta\overline{\varphi_j} = \overline{\Delta\varphi_j} = \lambda_j\overline{\varphi_j}$$

$\overset{\text{earlier}}{=} \hat{k}(\lambda_j)\,\overline{\varphi_j(w)}$

thm

where

$$\hat{k}(\lambda_j) = \int_{\mathbb{H}} \underbrace{k(z,i)\,\omega_{\lambda_j}(z;i)}\,d\mu(z)$$

spans the 1-dim. space of spherical fts.

We want to "take the trace"

$$\int_M K(z,z)\,d\mu(z)$$

$$= \sum_{j\geq 0} \hat{k}(\lambda_j) \underbrace{\int_M |\varphi_j(z)|^2\,d\mu(z)}_{=1}$$

This is sometimes called the pretrace formula

$$\int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in\Gamma} k(z,\gamma z)\, d\mu(z) = \dots$$

$\{\gamma\} = \{\tau\gamma\tau^{-1} : \tau\in\Gamma\}$

the Γ-conjugacy class of $\gamma\in\Gamma$.

Note that γ appears several times in $\{\gamma\}$ if we don't remove all τ's in

$$Z(\gamma) = \{\tau\in\Gamma : \tau\gamma = \gamma\tau\}$$

$$\dots = \sum_{\substack{\{\gamma\} \\ \gamma \text{ hyp.}}} \int_{\Gamma\backslash\mathbb{H}} \sum_{\tau\in Z(\gamma)\backslash\Gamma} k(z,\tau\gamma\tau^{-1}z)\, d\mu(z)$$

$$+ \int_{\Gamma\backslash\mathbb{H}} \underbrace{\underbrace{k(z,z)}_{k(\underbrace{d(z,z)}_{=0})}\, d\mu(z)}_{} = \dots$$

$= k(0)\,\text{area}(M)$

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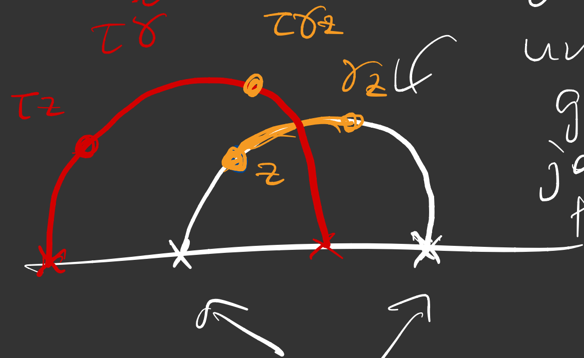
$$+ \sum_{\substack{\{\gamma\} \\ \gamma \text{ hyp}}} \int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma z)\, d\mu(z)$$

(once more by folding/unfolding)

<u>A description of a fundamental domain for $Z(\gamma)$</u>

γ hyp.



$\tilde{\gamma}$ the unique geodesic joining them

$Z(\gamma) = \{\tau\in\Gamma : \tau\gamma = \gamma\tau\}$

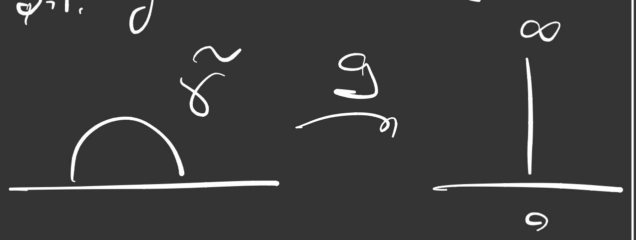
the two fixed points of γ

In general, γ does not fix $\tau\tilde{\gamma}$. (It will if $\tau\tilde{\gamma} = \tilde{\gamma}$)

Claim: $Z(\gamma) = \mathrm{Stab}_\Gamma(\tilde{\gamma})$.

Proof:

Choose $g \in G$ s.t. $g\tilde{\gamma} = i\mathbb{R}_{>0}$



$$\mathrm{Stab}_\Gamma(g\tilde{\gamma}) = \Gamma \cap A \cong \mathbb{Z}$$

$\llcorner$ diagonal matrices in G

$$= \{\gamma \in \Gamma : g^{-1}\gamma g \tilde{\gamma} = \tilde{\gamma}\}$$

$$= g\, \mathrm{Stab}_\Gamma(\tilde{\gamma})\, g^{-1}$$

$$\Rightarrow \quad \mathrm{Stab}_\Gamma(\tilde{\gamma}) = \langle \gamma_0 \rangle$$

$$\langle \gamma_0 \rangle \subset Z(\gamma)$$

$\gamma = \gamma_0^k$ for some $k \in \mathbb{N}$

hence $\gamma_0 \gamma = \gamma \gamma_0$.

$\tau \in Z(\gamma)$

$$\tau\tilde{\gamma} = \tau\gamma\tilde{\gamma} = \gamma\tau\tilde{\gamma}$$

$\Rightarrow$ γ fixes $\tau\tilde{\gamma}$

but $\tilde{\gamma}$ is the unique geodesic fixed by γ

$$\Rightarrow \quad \tilde{\gamma} = \tau\tilde{\gamma}$$

$\square$

This shows us that

$$Z(\gamma) = \langle \gamma_0 \rangle$$

where $\gamma = \gamma_0^k$ for some $k \in \mathbb{N}$.

def: We call $\gamma_0 \in \Gamma$ *primitive* if $\nexists\, \gamma \in \Gamma$ s.t. $\gamma_0 = \gamma^k$ for some $k \in \mathbb{N}$.

Before we go further, recall what we here

$$\int_M K(z,z)\,d\mu(z)$$

$$= k(0)\,\text{area}(M) + \sum_{\substack{\{\gamma\} \\ \gamma \text{ hyp}}} \int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma z)\,d\mu(z)$$

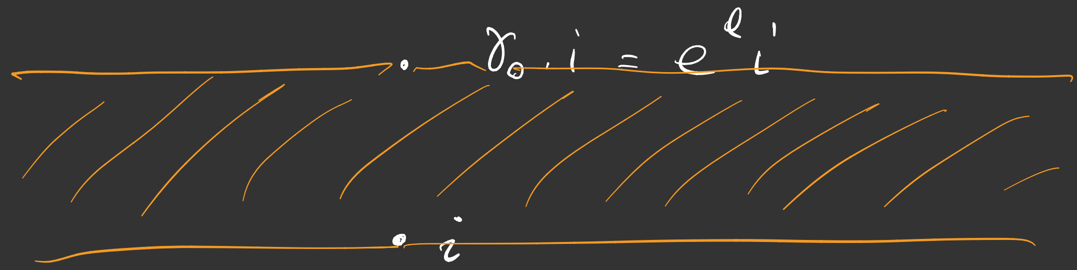
$$= k(0)\,\text{area}(M) + \sum_{\substack{\{\gamma_0\} \\ \gamma \text{ hyp. prim.}}} \sum_{n\geq 1} \int_{Z(\gamma_0)\backslash\mathbb{H}} k(z,\gamma_0^n z)\,d\mu(z)$$

Suppose γ_0 fixes $i\mathbb{R}_{>0}$.

We can parametrize γ_0 as

$$\gamma_0 = \begin{pmatrix} e^{\ell/2} & \\ & e^{-\ell/2} \end{pmatrix}$$

(where $\ell > 0$). Now a fundamental domain for $Z(\gamma_0)$ is



(We chose the above parametrization for γ_0 s.t.

$$d_{\mathbb{H}}(i, e^{\ell} i) = \ell$$

and ℓ is the length of the closed geodesic segment from i to $e^{\ell} i$.

The length ℓ is conjugacy-class invariant:

$$\cosh\left(\tfrac{\ell}{2}\right) = 1 + \frac{|e^{\ell/2}-1|^2}{2e^{\ell}}$$

$$= 1 + \frac{e^{\ell/2}+e^{-\ell/2}-2}{2} = \frac{|\operatorname{tr}\gamma_0|}{2}$$

$$\rightsquigarrow \quad \ell = 2\cosh^{-1}\frac{|\operatorname{tr}\gamma_0|}{2}.$$

<u>Prop</u>: If $\gamma = g^{-1}\gamma_0 g \quad (g \in G)$, then

$$\int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma^n z)\, d\mu(z) = \int_{Z(\gamma_0)\backslash\mathbb{H}} k(z,\gamma_0^n z)\, d\mu(z)$$

<u>Proof</u>: exercise.

This means we can always choose g s.t. $g\tilde{\gamma} = i\mathbb{R}_{>0}$ and

$$\int_{Z(\gamma_0)} k(z,\gamma_0^n z)\, d\mu(z) = \int_{-\infty}^{\infty}\int_{1}^{e^{\ell}} k(z, e^{n\ell}z)\,\frac{dy\,dx}{y^2}$$

$$\cosh(d(z,e^{n\ell}z)) = 1 + \frac{|z|^2\,|e^{n\ell/2}-e^{-n\ell/2}|^2}{2y^2}$$

$$= 1 + 2\sinh^2\left(\frac{n\ell}{2}\right)\frac{x^2+y^2}{y^2}$$

$$k(z,e^{n\ell}z) = U(\cosh d(z,e^{n\ell}z))$$

$$\int_{-\infty}^{\infty}\int_{1}^{e^{\ell}} k(z, e^{n\ell} z)\,\frac{dy\,dx}{y^2}$$

$$= \int_1^{e^{\ell}} \frac{dy}{y} \int_{-\infty}^{\infty} u\left(1+2\sinh^2\left(\frac{n\ell}{2}\right)(x^2+1)\right) dx$$

$$= \frac{\ell}{\sinh\frac{n\ell}{2}} \int_{\sinh^2\frac{n\ell}{2}}^{\infty} \frac{u(1+2u)}{\sqrt{u-\sinh^2\frac{n\ell}{2}}}\, du$$

$$= \frac{\ell}{\sqrt{2}\sinh\frac{\ell n}{2}} \int_{n\ell}^{\infty} \frac{k(\rho)\sinh(\rho)\, d\rho}{\sqrt{\cosh\rho - \cosh(n\ell)}}$$

$$= \frac{\ell}{2\sinh\left(\frac{\ell n}{2}\right)} \cdot g(n\ell)$$

where g is the Fourier inverse of the Selberg transform

Selberg transform

$$h(r) = \hat{k}\left(\frac{1}{4}+r^2\right)$$

$$h(r) = \hat{g}(r)$$

Fourier transform

Our pretrace formula has evolved into

$$\sum_{j\geq 0} \hat{k}(\lambda_j) = \operatorname{area}(M)\, k(0) + \sum_{\substack{\{\gamma\} \\ \gamma \text{ prim hyp}}} \frac{\ell}{2} \sum_{n\geq 1} \frac{g(n\ell)}{\sinh\left(\frac{n\ell}{2}\right)}$$

Using the inverse Selberg transform, one can write

$$k(0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\cdot t\cdot \tanh(t)\,dt$$

<u>Thm</u>: (Selberg trace formula)

M compact hyperbolic surface, $g \in C_c^\infty(\mathbb{R})$ even, $h = \hat{g}$ its Fourier transform. Then

$$\sum_{j\geq 0} h(r_j) = \frac{\operatorname{area}(M)}{4\pi}\int_{-\infty}^{\infty} h(t)\, t\cdot \tanh(t)\,dt + \sum_{\substack{\{\gamma\}\\ \gamma \text{ prim hyp}}} \frac{\ell}{2} \sum_{n\geq 1} \frac{g(n\ell)}{\sinh\left(\frac{n\ell}{2}\right)}$$

where the sum on the LHS is indexed over $\lambda_j = \frac{1}{4} + r_j^2$ eigenvalues of Δ, and ℓ is the translation length of γ.

Why the parametrization $\lambda_j = \frac{1}{4} + r_j^2$?

$\mathbb{H} \to \mathbb{H}$

$z \mapsto y^s$

$(s \in \mathbb{C})$

$\Delta y^s = s(1-s)\,y^s$

$s = \frac{1}{2} + ir \quad (r \in \mathbb{C})$

$$\lambda = s(1-s) = \frac{1}{4} + r^2$$

Since Δ is symmetric, its eigenvalues are in $[0,\infty)$.

This forces either

$r \in \mathbb{R}$

$\operatorname{Re}(s) = 1/2$

$\lambda = s(1-s) \in [\frac{1}{4}, \infty)$

or

$r = it$

$|t| < 1/2$

$\lambda \in [0, 1/4)$

Plan for next class

Length spectrum = { collection of lengths of closed oriented geodesics on M, ordered by size }

Thm (Huber)

Two cpt. hyperbolic surfaces M and M' have the same spectrum of the Laplacian if and only if they have the same length spectrum.

Thm: $0 = \lambda_0 < \lambda_1 \leq \dots \to \infty$ spectrum of the Laplacian, where eigenvalues appear according to their multiplicity.

$$N(\lambda) = \#\{j \geq 0 : \lambda_j \leq \lambda\}$$

$$N(\lambda) \sim \frac{\text{area}(M)}{4\pi} \lambda^2 \quad (\text{as } \lambda \to \infty)$$

Notation:

$f(x) \sim g(x) \quad x \to \infty$

$$\iff \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

This last Thm. is called Weyl's law.

The original Weyl's law (1911) says:

$\Omega \subset \mathbb{R}^2$ bdd. measurable domain

Dirichlet problem:

$$\begin{cases} \Delta u = \lambda u & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

Again by spectral thm,

$$0 = \lambda_0 < \lambda_1 \leq \cdots \longrightarrow \infty$$

$$N(\lambda) \sim \frac{\text{area}(\Omega)}{4\pi} \lambda^2 \quad \text{as } \lambda \to \infty.$$

If M is a compact hyperbolic surface, then

$$\text{area}(M) = 4\pi(g-1)$$

where g is the genus of M.

Coro (of Weyl's law):

Two cpt. hyp. surfaces M and M' are isospectral only if they have the same genus.