M compact hyperbolic surface

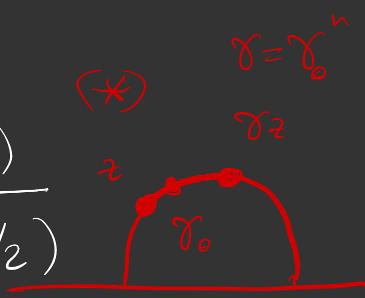
$g \in C_c^{\infty}(\mathbb{R})$ even

$$h(r) = \hat{g}(r) = \int_{-\infty}^{\infty} g(u)\, e^{iru}\, du$$

its Fourier transform.

$$\sum_{j \geq 0} h(r_j) = \frac{\text{area}(M)}{4\pi} \int_{-\infty}^{\infty} h(r)\, r \tanh(\pi r)\, dr$$

$$+ \sum_{\substack{\{\gamma\} \\ \gamma \text{ hyp.}}} \frac{\ell_0\, g(\ell)}{2 \sinh(\ell/2)}$$



where the sum on the LHS ranges over $\lambda_j = \frac{1}{4} + r_j^2$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \qquad \lambda_j \longrightarrow \infty$$

the spectrum of $\Delta\big|_{L^2(M)}$.

<u>Selberg's trace formula</u>

<u>Today</u>:

<u>Huber's theorem</u>

Two compact hyp. surfaces have the same Laplacian spectrum iff they have the same length spectrum.

<u>Weyl's law</u>: As $\lambda \to \infty$,

$$\#\{j \geq 0 : \lambda_j < \lambda\} \sim \frac{\text{area}(M)}{4\pi} \lambda$$

(*)

$$\sum_{\substack{\{\gamma\} \\ \gamma \text{ prim. hyp.}}} \sum_{n \geq 1} \frac{\ell\, g(n\ell)}{2 \sinh\left(\frac{n\ell}{2}\right)}$$

$\mathcal{C} = \{$all closed oriented geodesics on $M\}$

$\mathcal{L}$ = the corresponding set of lengths, ordered by ascending size.

This is what is called the <u>length spectrum</u>, and it accounts for each length with multiplicity.

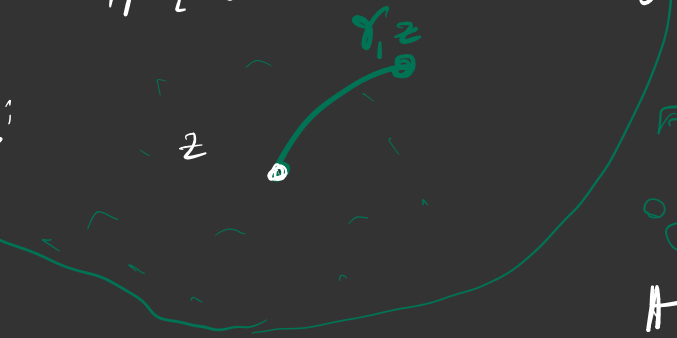
It is a discrete set.

In fact:

<u>Prop</u>:

$$\pi(L) = \#\{\ell \in \mathcal{L} : \ell \leq L\} = O(e^L)$$

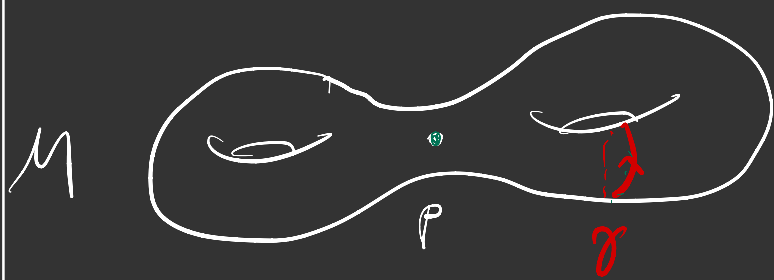
<u>Proof</u>:



hyp-ball of radius $L + 2\operatorname{diam} M$

$\mathbb{H}$

$z \mapsto p$



l_{γ_1} = length of γ_1

$\leq l_\gamma + 2\operatorname{diam}(M)$

$\leq L + 2\operatorname{diam} M$.

Take γ_1 a closed geodesic on M passing through p

$\gamma_1 \sim \gamma$ (free homotopy)

Take all $\gamma \in \mathcal{C}$ s.t. $l_\gamma \leq L$

Proof of Huber's thm.

Idea: Apply Selberg's trace formula to the heat kernel.

Its Selberg transform is

$$h(r) = e^{-tr^2} \quad (t>0)$$

$$\sum_{j\geq 0} e^{-tr_j^2} = \sum_{j\geq 0} e^{-\lambda_j t} e^{t/4}$$

is the LHS of the STF, so that we rewrite it as

$$\sum_{j\geq 0} e^{-t\lambda_j} = \frac{\operatorname{area}(M)}{4\pi} e^{-t/4} \cdot \int_{-\infty}^{\infty} e^{-tr^2} r \tanh(\pi r)\, dr$$

$$+ e^{-t/4} \sum_{\gamma \in \mathcal{C}} \frac{l_0\, g(\ell)}{2\sinh(\ell/2)}$$

The number of lifts $\gamma_i z \in \mathbb{H}$ is $\leq \# \{ \gamma \in \Gamma : d(\gamma z, z) < L + 2\operatorname{diam} M \}$

$\lesssim$ area (hyperbolic ball of radius $L + 2\operatorname{diam} M$)

$= C e^{L}$ ∎

where $g(\ell) = \dfrac{1}{\sqrt{4\pi t}} e^{-\ell^2/4t}$

The LHS of this last equation is the spectral partition function for $\Delta_{\mathbb{H}^2(M)}$. From the spectral partition ft., we may recover each single eigenvalue and its multiplicity.

μ_j = mult. of $\tilde{\lambda}_j$

$$\sum_{j\geq 0} e^{-t\lambda_j} = \sum_{j\geq 0} \mu_j e^{-t\tilde{\lambda}_j}$$

$0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \dots$

spectrum w/o multiplicities

$$\lim_{t\to\infty} e^{t\omega} \sum_{j\geq 0} \mu_j e^{-t\tilde{\lambda}_j} =$$

$$= \lim_{t\to\infty} \sum_{j\geq 0} \mu_j e^{t(\omega-\tilde{\lambda}_j)} = \begin{cases} 0 & \omega<\tilde{\lambda}_0 \\ \mu_0 & \omega=\tilde{\lambda}_0 \\ \infty & \omega>\tilde{\lambda}_0 \end{cases}$$

and so on.

Note

$$\sigma(t) = \left| \int_{-\infty}^{\infty} e^{-tr^2} \cdot r \cdot \tanh(\pi r)\, dr \right|$$

$$\leq 2\int_0^{\infty} e^{-tr^2} r\, dr$$

$$= \frac{e^{-tr^2}}{-t}\Big|_0^{\infty} = \frac{1}{t}$$

<u>Exercise</u>: By integration by parts

$$\sigma(t) \sim \frac{1}{t}$$

Once we have recovered all eigenvalues (and their mult.) up to $1/4$, multiply the TF by $\frac{e^{t/4}}{\sigma(t)} \cdot 4\pi$:

$$\frac{e^{t/4}}{\sigma(t)} \sum_{\tilde{\lambda}_j > 1/4} \mu_j e^{-t\tilde{\lambda}_j} - \text{area}(M)$$

= a function we know

Taking $t\to\infty$, we recover area(M). Continue with the eigenvalues above $1/4$.

This shows how to recover the spectrum of the Laplacian from the length spectrum

In the other direction: suppose we knew the spectrum of the Laplacian. To recover the length spectrum,

$$\frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{\ell \in \mathcal{L}} \frac{\ell_0 e^{-\ell^2/4t}}{2\sinh(\ell/2)} \cdot e^{\omega^2/4t}$$

and take $t \to 0$ to recover each distinct length in $\mathcal{L}$ with its multiplicity.

<u>Claim</u>: $\displaystyle\sum_{j \geq 0} e^{-t\lambda_j} \sim \frac{\operatorname{area}(M)}{4\pi t}$

as $t \to 0^+$.

This says that the spectrum of Laplacian also determines the area. Once this claim is proved, we are done.

Rewrite length spectrum contribution as

$$\sum_{\ell \in \mathcal{L}} \psi(\ell) = \lim_{L \to \infty} \sum_{\ell \leq L} \psi(\ell)$$

<u>Abel summation</u>:

If $\psi \in C^1$,

$$\sum_{\ell \leq L} \psi(\ell) = \pi(L)\psi(L) - \int_0^L \pi(u)\, d\psi(u)$$

where

$$\pi(L) = \#\{\ell \in \mathcal{L} : \ell \leq L\}$$

as before

Using Abel summation, you can check for yourselves that

$$\lim_{L\to\infty} \sum_{\ell \leq L} \Psi(\ell) < \infty$$

if $\Psi(L) = O\left(\frac{e^{-L}}{L \cdot \log L}\right)$

Exercise:

$$\Psi(L) = O\left(\frac{e^{-L}\, e^{-c/t}}{L \cdot \log L \cdot \sqrt{t}}\right)$$

where $c > 0$ is a small constant.

Once we have proven this we have that

$$\sum_{\ell \in \mathcal{L}} \Psi(\ell) \longrightarrow 0 \quad \text{as } t \to 0$$

Hence, letting $t \to 0$ in the STF, we are left with

$$\sum_{j \geq 0} e^{-\lambda_j t} = \frac{\text{area}(M)}{4\pi}\left(\frac{1}{t} + O(1)\right)$$

This proves the claim. $\square$

The last part of this proof shows that we can deduce the area(M) from the Laplacian spectrum.

Weyl's law: As $\lambda \to \infty$

$$\#\{j \geq 0 : \lambda_j \leq \lambda\} \sim \frac{\text{area}(M)}{4\pi} \lambda$$

Hardy - Littlewood Tauberian thm.

$(a_n \geq 0)$

$$\sum_{n \geq 0} a_n x^n \sim \frac{1}{1-x} \qquad |x|<1$$

as $x \to 1^-$

$$\Rightarrow \sum_{n \leq N} a_n \sim N \quad \text{as } N \to \infty$$

Proof of Weyl's law:

Apply the HL Tauberian thm to

$$\sum_{j \geq 0} e^{-\lambda_j t} = \frac{\text{area}(M)}{4\pi}\left(\frac{1}{t} + O(1)\right)$$

$$x = e^{-t} \to 1^- \iff t \to 0^+$$

Since

$$\sum_{j \geq 0} \frac{4\pi}{\text{area}(M)} e^{-\lambda_j t} \sim \frac{1}{1-e^{-t}} \sim \frac{1}{t},$$

$$\Rightarrow \sum_{\lambda_j \leq \lambda} \frac{4\pi}{\text{area}(M)} \sim \lambda$$

as $\lambda \to \infty$

$\square$

Prop: If M is a compact hyperbolic surface,

$$\text{area}(M) = 4\pi(g-1)$$

where g is the genus.

Proof:

- Topological classification of compact orientable topological surfaces: Each such surface is homeomorphic to

$g=0$ $\quad$ $g=1$ $\quad$ $g\geq 2$

- Gauss–Bonnet theorem: If M is a compact Riemannian surface,

$$\int_M K\,dA = 2\pi\,\chi(M)$$

K: Gaussian curvature $\quad$ $\chi(M)$: Euler characteristic

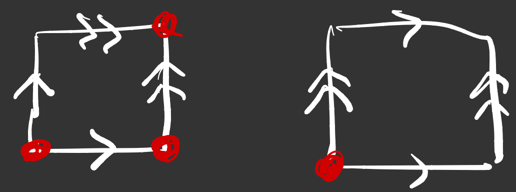
Since M is compact hyperbolic

$$K \equiv -1$$

and so Gauss–Bonnet is

$$\text{area}(M) = -2\pi\,\chi(M)$$

- The Euler characteristic is a topological invariant



$3-2+1 = 2$ $\quad$ $1-2+1 = 0$

- take 2 copies of torus
- cut out a disk
- glue along the boundaries

$$\chi = V - E + F$$

Exercise: $\chi = 2 - 2g$

We have seen that: if M, M' are two hyp. cpct surfaces with the same

$\Rightarrow \text{area}(M) = -2\pi(2-2g)$
$= 4\pi(g-1)$

Laplacian spectrum, they are topologically equivalent.

<u>Question</u>: Are they geometrically equivalent, i.e. isometric?

This is false. First counterexample due to Milnor (~1968)

"Can one hear the shape of a drum?"

wave equation:
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -c\Delta u & \text{on } \Omega \subset \mathbb{R}^2 \quad (\text{for some constant } c>0) \\ u\big|_{\partial\Omega} = 0 \end{cases}$$