$z, w \in \mathbb{H}$, Γ a Fuchsian gp, $M = \Gamma \backslash \mathbb{H}$

$$N_R = \#\{ z' \in \Gamma z : d(w, z') < R \}$$

$$= \sum_{\gamma \in \Gamma} 1_R(d(w, \gamma z))$$

$$\text{"="} \sum_{j \geq 0} h_R(r_j) \varphi_j(z) \overline{\varphi_j(w)}$$

$$= \frac{\text{area } B_R}{\text{area } M} + \sum_{j \geq 1} h_R(r_j) \varphi_j(z) \overline{\varphi_j(w)}$$

As usual:

- area $B_R \sim e^R$ $(R \to \infty)$
- $1_R(t) = \begin{cases} 1 & \text{if } t < R \\ 0 & t \geq R \end{cases}$
- $\lambda_j = \frac{1}{4} + r_j^2$ (with $r_j \in \mathbb{C}$)
- h_R the Selberg transform of 1_R.

$\lambda_j \in [0, 1/4)$

- $r_j \in [-\frac{i}{2}, \frac{i}{2}]$
- There can only be finitely many $\lambda_j \in [0, 1/4)$ (by discreteness.
- $\frac{1}{2} = |r_0| > |r_1| \geq |r_2| \geq \dots \geq |r_k| > 0$

$\lambda_j \geq 1/4$

- $r_j \in \mathbb{R}$
- $0 \leq |r_{k+1}| \leq |r_{k+2}| \leq \dots$

Computing the h_R for 1_R,

$$N_R = \frac{\text{area}(B_R)}{\text{area}(M)} + \sum_{\lambda_j \in (0, 1/4)} c_j e^{R(\frac{1}{2} + |r_j|)} + O\left(\sum_{\lambda_j \geq 1/4} \frac{1}{|r_j|^{3/2}} \cdot e^{R/2} \right) + O(e^{R/2})$$

$c_j \in \mathbb{C}$ depending on $|r_j|$, z, w, NOT on R

If $\Sigma < \infty$, then

(1) $$N_R \sim \frac{\text{area}(B_R)}{\text{area}(M)} \quad \text{as } R \to \infty.$$

and if $\lambda_1 < 1/4$, then

(2) $$\left| \frac{N_R}{\text{area}(B_R)} - \frac{1}{\text{area}(M)} \right| \ll e^{R(-\frac{1}{2}+|r_1|)}$$

and if instead $\lambda_1 \geq 1/4$

(3) $$\left| \cdots \right| \ll e^{-R/2}$$

This shows that λ_1 (at least if $\lambda_1 < 0$) has some geometric significance

For comparison, $\lambda_0 \longleftrightarrow \text{area}(M)$

On the other hand, this example shows that we would like to have

$$\lambda_1 \geq 1/4.$$

Rmk: $\lambda_1 = \lambda_1 - \lambda_0$ is called the spectral gap.

Ideally, there would be a uniform spectral gap for all Fuchsian groups. However we'll show

Thm: $\forall \varepsilon > 0$ $\exists$ a compact hyperbolic surface M s.t.

$$\lambda_1(M) < \varepsilon.$$

However

$$\sum_{\lambda_j \geq 1/4} \frac{1}{|r_j|^{3/2}} = \sum_{\lambda_j \geq 1/4} \frac{1}{(\lambda_j - 1/4)^{3/4}}$$

$$> \sum_{\lambda_j \geq 1/4} \frac{1}{\lambda_j^{3/4}}$$ does not converge

$$\sum_{1/4 \leq \lambda_j < T} \frac{1}{\lambda_j^{3/4}} > T^{-3/4} \Big(\sum_{\frac{1}{4} \leq \lambda_j < T} 1 \Big) \xrightarrow{\text{if } T \to \infty} \infty$$

$$\underset{\text{Weyl's law}}{\sim} c \cdot T$$

In practice, one replaces 1_R by a smooth approximation to obtain better estimates for the Selberg transform.

If we do that, we eventually recover (1) and (2), but it comes at the cost of a weaker error term for (3), namely

$$|\dots\dots| \ll e^{-R/3}$$

<u>Conjecture</u>: the expected rate $e^{R(\varepsilon - 1/2)}$ (for any small $\varepsilon > 0$)

Prop: Let M be a compact hyperbolic surface. Then

$$\lambda_1(M) = \inf \frac{\int_M |\nabla f|}{\int_M |f|^2}$$

is over all functions $f \in \mathrm{Dom}(\Delta) \cap (\mathbb{C} \cdot 1)^{\perp}$.

Rmk: $L^2(M) = \mathbb{C} \cdot 1 \oplus (\mathbb{C} \cdot 1)^{\perp}$

Exercise: Show that

$$(\mathbb{C} 1)^{\perp} = \{ f : \int_M f = 0 \}$$

... Then

$$\langle \Delta f, f \rangle = \lambda_1 \|f\|^2 \overset{\mathrm{IBP}}{=} \int_M |\nabla f|^2$$

$\square$

Proof: Let $f \in \mathrm{Dom}(\Delta) \cap (\mathbb{C} 1)^{\perp}$.

In particular f admits the spectral expansion

$$f(z) = \sum_{j \geq 1} \langle f, \varphi_j \rangle \varphi_j(z)$$

We can check that

$$\int_M |f|^2 = \sum_{j \geq 1} |\langle f, \varphi_j \rangle|^2$$

and

$$\int_M |\nabla f|^2 = \sum_{j \geq 1} \lambda_j |\langle f, \varphi_j \rangle|^2$$

$$\geq \lambda_1 \sum_{j \geq 1} |\langle f, \varphi_j \rangle|^2 = \int_M |f|^2.$$

Take now f to be an eigenfunction for λ_1

$\Delta f = \lambda_1 f$ .....

Thm: $\forall \varepsilon > 0$ $\exists$ a cpt. hyperbolic surface M s.t. $\lambda_1(M) < \varepsilon$.

Sketch of proof #1 (after Cheeger)

Let M be a compact hyperbolic surface. Choose a closed geodesic γ on M s.t $M \setminus \gamma$ is still connected. Recall:

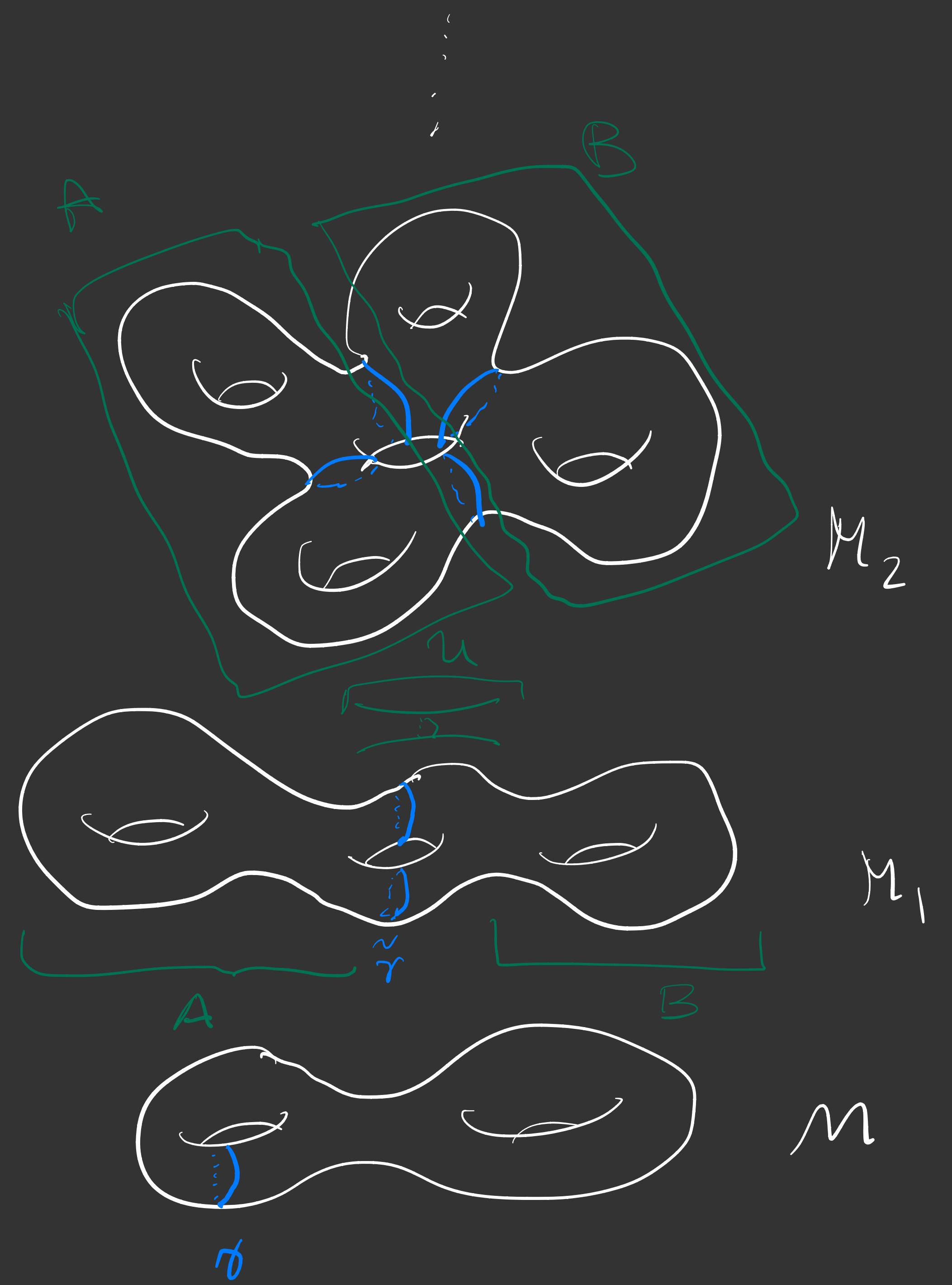
$$\lambda_1(M_i) = \inf \frac{\int_M |\nabla f|^2}{\int_M |f|^2}$$

Choose on each M_i a function f_i s.t.

$$f_i = \begin{cases} 1 & \text{on } A \\ -1 & \text{on } B \\ \text{linear transition} & \text{on small pipes in nbhd. of } \tilde{\gamma} \end{cases}$$

Observe

$$\int_{M_i} f_i = 0$$



$$\int_{M_i} |\nabla f_i|^2 \text{ bdd}$$

$$\int_{M_i} |f_i|^2 \approx \text{area}(M_i) \approx Li \cdot \text{area}(M)$$

This shows that

$$\lambda_1(M_i) \longrightarrow 0 \quad \text{as } i \to \infty$$

$\square$

Morally:

$\lambda_1 \geq 1/4 \quad \leftrightsquigarrow \quad M$ "short and fat"

On the contrary, to obtain the example above, we constructed a sequence of compact hyp. surface that is every time longer and thinner.

Cheeger constant

$$h(M) = \inf \frac{\text{area}(\partial D)}{\text{area}(D)}$$

where inf is taken over all bdd. regular domains $D \subset M$ s.t. $\text{area}(D) \leq \frac{1}{2}\text{area}(M)$.

Rmk: Definition holds more generally for any compact Riemannian manifold.

$$\frac{h(M)^2}{4} \leq \lambda_1(M) \leq C\left(h(M)^2 + h(M)\right)$$

Cheeger's inequality (1970)

Buser (early 80s)

(for some constant)

Idea of proof #2, after Selberg

Let M be a compact hyperbolic surface. We consider the following modified spectral problem: find solutions to

$$\begin{cases} \Delta f = \lambda f \\ \int_M |f|^2 < \infty \\ f(\gamma z) = \chi(\gamma) f(z) \quad \forall \gamma \in \Gamma, \ M = \Gamma \backslash \mathbb{H} \end{cases}$$

where $\chi : \Gamma \to S^1$ homomorphism.

Fix $\Theta \in \mathbb{T}^{2g}$, and consider the associated character χ_Θ.

We are going to use the

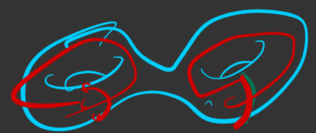
Aside: The set of all characters i.e. homomorphisms from Γ to S^1, forms a group

$$\hat{\Gamma} = \mathrm{Hom}(\Gamma, S^1)$$

$$= \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], S^1)$$

$$[\Gamma,\Gamma] = \{aba^{-1}b^{-1} : a, b \in \Gamma\}$$

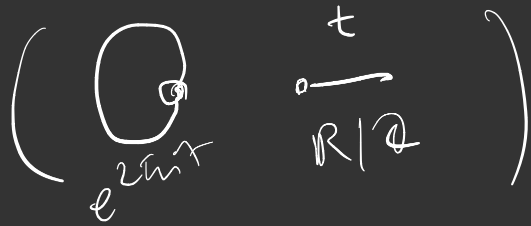
$$\cong \mathrm{Hom}(H_1(M), S^1)$$



$$\cong \mathrm{Hom}(\mathbb{Z}^{2g}, S^1)$$

where g is the genus of M

$$\cong \mathrm{Hom}(\mathbb{Z}^{2g}, \mathbb{T}) \cong \mathbb{T}^{2g}$$



following facts:

1) The spectral problem for χ_Θ admits a complete resolution

$$0 \leq \lambda_0(\Theta) \leq \lambda_1(\Theta) \leq \cdots\cdots$$

2) $\lambda_0(\Theta)$ is continuous in Θ

This means for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|\Theta| < \delta$, $\lambda_0(\Theta) < \varepsilon$

3) $\lambda_0(\Theta) = 0$ iff $\Theta = 0$

Pf: $\Leftarrow$ if $\Theta = 0$, $\chi_\Theta \equiv 1$ (the trivial character) and we recover our usual spectral problem whereby the bottom of the spectrum is $\lambda_0 = 0$.

Bottom line: $\mathbb{T}^{2g}$ parametrizes the characters on Γ

$\Rightarrow$

$$\lambda_0(\Theta) = \inf \frac{\int_M |\nabla f|^2}{\int_M |f|^2} = 0$$

over all $f \in \mathrm{Dom}(\Delta)$

$\Rightarrow$ $\exists$ an eigenfct. for $\lambda_0(\Theta)$ with $|\nabla f| = 0$ $\Rightarrow$ $f = \text{constant}$

Since $\underbrace{f(\gamma z)}_{\text{constant}} = \chi_\Theta(\gamma) \underbrace{f(z)}_{\text{constant}}$ for any $\gamma \in \Gamma$, $\chi_\Theta \equiv 1$ $\Longleftrightarrow$ $\Theta = 0$

Fix $\varepsilon > 0$. Choose $\Theta \in \mathbb{T}^{2g}$ s.t. $|\Theta| < \delta$ and $\Theta \in (\mathbb{Q}/\mathbb{Z})^{2g}$.

Taking the corresponding character $\chi_\Theta : \Gamma \to S^1$ has finite image. <u>Exercise</u>: Convince yourselves that the image lies in ζ_N - the Nth roots of unity.

Define

$$\Gamma_\Theta = \ker(\chi_\Theta) \underset{\text{index}}{\overset{\text{finite}}{<}} \Gamma.$$

If f is a solution of the modified spectral problem for $\lambda_0(\Theta)$

Geometrically:

$$\boxed{\begin{array}{c} \Gamma_\Theta \backslash \mathbb{H} \\ \downarrow \\ \Gamma \backslash \mathbb{H} \end{array}}$$

namely

$$\Delta f = \lambda_0(\Theta) f$$
$$f(\gamma z) = \chi_\Theta(\gamma) f(z) \quad \forall \gamma \in \Gamma$$

then f is Γ_Θ-invariant

$\Rightarrow \lambda_0(\Theta) < \varepsilon$ is an eigenvalue for the standard spectral problem on $\Gamma_\Theta \backslash \mathbb{H}$.