$\Gamma(N) = \ker\left(SL_2\mathbb{Z} \to SL_2\mathbb{Z}_N\right)$ principle congruence groups

$\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$

$X_N = \Gamma(N)\backslash\mathbb{H}$ noncompact hyperbolic surfaces

Recall:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \iff \begin{array}{l} a \equiv d \equiv 1 \ (N) \\ b \equiv c \equiv 0 \ (N) \end{array}$$

$$SL_2\mathbb{Z} = \Gamma(1)$$

$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$ hence X_N are noncompact

Selberg's eigenvalue conjecture

(1965) $\lambda_1(X_N) \geq 1/4$ for each $N \geq 1$

Remarks:

- Selberg proved in the same 1965 article that $\lambda_1(X_N) \geq 3/16$
- Booker–Strömbergsson (2007) verified the conjecture up to $N < 857$.

Plan: Describe

1. what X_N look like;
2. what the spectrum of a noncompact hyperbolic surface looks like;
3. relation with expander graphs;
4. Brooks' assertion: X_N are short, fat, and have

(1) **Prop**: Let $\Gamma < SL_2\mathbb{R}$ be a Fuchsian group, with fundamental domain $\mathcal{F}$. Let $\Gamma' < \Gamma$ be a sgp of finite index. Then

$$\mathcal{F}' = \bigsqcup \bar{\gamma}_i \mathcal{F} \quad (*)$$

is a fundamental domain for Γ', where $\Gamma = \bigcup \Gamma' \gamma_i$

$\bar{\gamma}_i$ is the image of γ_i under the standard projection

$$SL_2\mathbb{R} \to PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm\}$$

Proof: exercise.

$\mathcal{F}$: the std. fund. dom. for $SL_2\mathbb{Z}$

Recall that $SL_2\mathbb{Z}$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

interesting symmetries.

Rmk: The union in $(*)$ is a union of $[\bar{\Gamma} : \bar{\Gamma}']$ copies of $\mathcal{F}$.

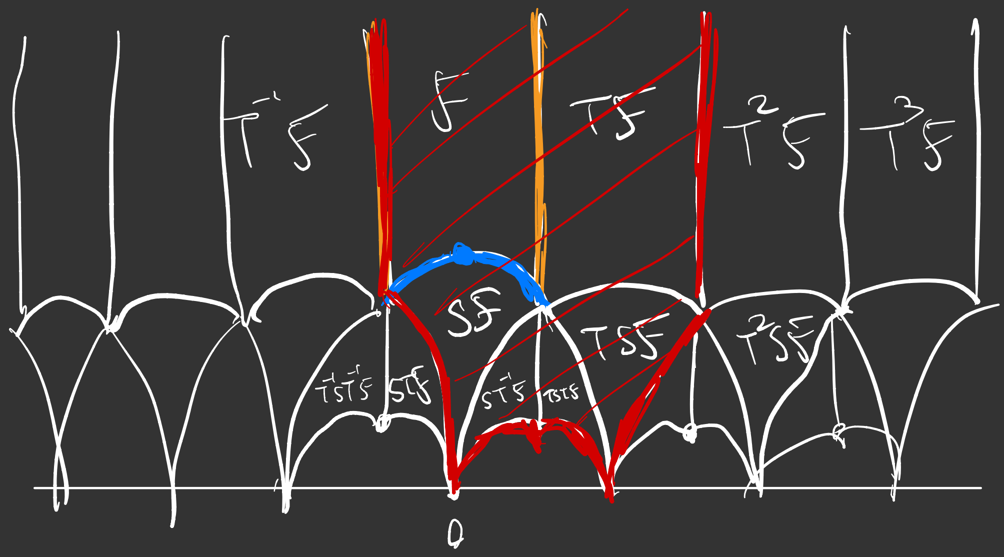
$\bar{\Gamma}, \bar{\Gamma}'$ are the images of Γ and Γ' (resp.) under the std. proj. $SL_2\mathbb{R} \to PSL_2\mathbb{R}$.

Here one needs to be a bit careful as it is not always true that

$$[\Gamma : \Gamma'] \neq [\bar{\Gamma} : \bar{\Gamma}']$$

e.g. $\Gamma = SL_2\mathbb{Z}$, $\Gamma' = \Gamma(N)$

$$\left| SL_2\mathbb{Z}/\Gamma(N) \right| = 2 \left| PSL_2\mathbb{Z}/\overline{\Gamma(N)} \right|$$



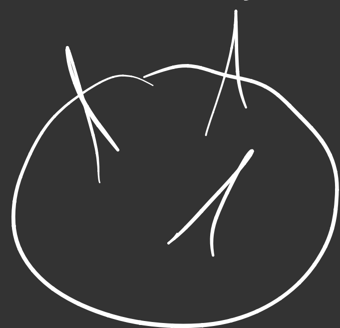
if $N \geq 3$

why: $-I \in SL_2 \mathbb{Z}$

$\notin \Gamma(N)$ if $N \geq 3$

$\Gamma(2) \rightsquigarrow [\overline{\Gamma(1)} : \overline{\Gamma(2)}] = [\Gamma(1) : \Gamma(2)] = |SL_2(\mathbb{Z}_2)| = 6$

Note $S, T \notin \Gamma(2)$. So X_2 (by gluing the edges)

is  (a sphere with three cusps)

More generally,

X_N looks like

"hedgehog shaped"

Cor: Keeping the notation as in the previous prop, we have

$$\text{area}(\mathcal{F}') = [\bar{\Gamma} : \bar{\Gamma}'] \cdot \text{area}(\mathcal{F})$$

Proof: Exercise.

Prop: $[SL_2\mathbb{Z} : \Gamma(N)] = |SL_2\mathbb{Z}_N|$

$$= N(N^2-1),$$

if $N = p$ prime.

Proof:

$SL_2(\mathbb{Z}_p) = \ker(\det : GL_2\mathbb{Z}_p \to \mathbb{Z}_p^*)$

$$|SL_2\mathbb{Z}_p| = \frac{|GL_2\mathbb{Z}_p|}{p-1}$$

$|GL_2\mathbb{Z}_p| = \#\{$ordered bases for the vector space $\mathbb{Z}_p \oplus \mathbb{Z}_p$ over $\mathbb{Z}_p\} = (p^2-1)(p^2-1-(p-1))$

$$= (p^2-1)\,p(p-1) \qquad \square$$

In particular,

$$\text{area}(X_p) = \frac{p(p^2-1)}{2} \cdot \frac{\pi}{3}$$

$$\sim \frac{\pi}{6} p^3 \quad (p \to \infty)$$

which you can compare to the examples we constructed last week

$\text{area}(M_i) = 2i \cdot \text{area}(M$

2 Recall:

The spectrum σ of $\Delta|_{L^2(\mathbb{R}^n)}$ is $\sigma = [0, \infty)$

For $\Delta|_{L^2(\mathbb{T}^n)}$,

$$\sigma = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$$

In the hyperbolic context,

$$\Delta|_{L^2(\mathbb{H})} \rightsquigarrow \sigma = [\tfrac{1}{4}, \infty)$$

If $M = \Gamma \backslash \mathbb{H}$ is a compact hyperbolic surface

$$\Delta|_{L^2(M)} \rightsquigarrow \sigma = \{0 = \lambda_0 < \lambda_1 \leq \dots\}$$

If now M is noncompact then the spectrum of $\Delta|_{L^2(M)}$ is

$$\sigma = \{0\} \cup \overbrace{\{0 < \lambda_1 \leq \lambda_2 \leq \dots\}}^{\text{cuspidal spectrum}} \cup \underbrace{[\tfrac{1}{4}, \infty)}_{\text{continuous spectrum}}$$

There is a conjecture of Phillips – Sarnak that says that for "generic" hyperbolic surfaces, the cuspidal spectrum is empty

It's a theorem of Selberg that the cuspidal spectrum is nonempty for X_N.

③

def: Let $X=(V,E)$ be a finite ($|V|<\infty$) graph. Its Cheeger constant

$$h(X)=\inf \frac{|\partial D|}{|D|}$$

where $D\subset V$ s.t. $0<|D|\leq \frac{|V|}{2}$

and ∂D is the set of edges that connect D to $V\setminus D$.

"short and fat"

$\longleftrightarrow$ "$|D|$ small, $|\partial D|$ large"

$\longleftrightarrow$ "sparse and highly connected"

This is the colloquial definition of an expander graph __def__: Let $(X_m)_{m\geq 1}$ be a family of finite connected(*) k-regular(**) graphs s.t.

$$|V_m| \longrightarrow \infty \quad \text{as} \quad m \to \infty.$$

We say that $(X_m)_{m\geq 1}$ is a __family of expanders__ if

$\exists\, \varepsilon > 0$ s.t. $h(X_m) > \varepsilon$ for each $m \geq 1$

(*) For each pair of vertices, there is an edgepath

(**) Each vertex is connected to k other vertices

__def__: Let G be a group. Let $S \subset G$ be a nonempty finite symmetric subset $(S = S^{-1})$ of G. The Cayley graph $\mathcal{G}(G, S)$ is the graph with vertex set

$$V = G$$

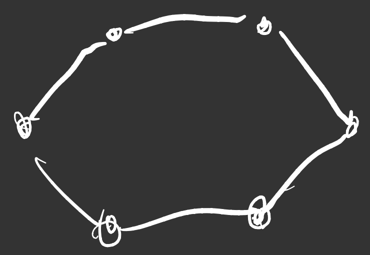
and edge set

g ——— h iff $\exists\, s \in S$ $g = sh$

Rmk:
S symmetric $\longleftrightarrow$ Cayley graph is undirected

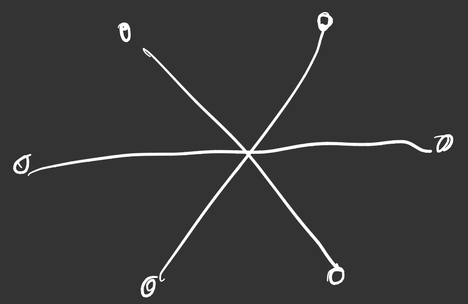
Examples:

$G = \mathbb{Z}_6$, $S = \{1, 5\}$

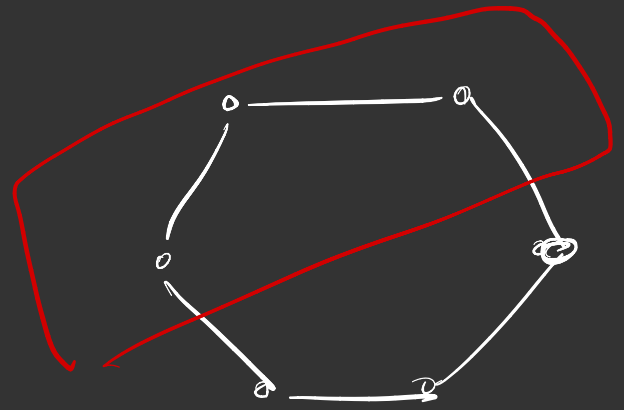


$\mathcal{G}(G, S) = C_6$

$G = \mathbb{Z}_6$, $S = \{3\}$



$G = SL_2(\mathbb{Z}_2)$ $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$



D

(although $\mathbb{Z}_6 \not\cong SL_2(\mathbb{Z}_2)$)

$$h(C_6) = \frac{2}{3}$$

$$h(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor} \approx \frac{4}{n}$$

Thm: Let Γ be a finitely generated Fuchsian group, let $S \subset \Gamma$ s.t. $S = S^{-1}$, $|S| < \infty$ and $\langle S \rangle = \Gamma$. Let $\{\Gamma_i\}$ be a family of normal finite-index sgps of Γ. Then

$$\mathcal{G}_i = \mathcal{G}(\overline{\Gamma}/\overline{\Gamma}_i, S)$$

is a family of expanders if and only if $\exists \varepsilon > 0$ s.t. $\lambda_1(\Gamma_i \backslash \mathbb{H}) \geq \varepsilon$ for each $i \geq 1$

Coro: The graphs

$$\left(\mathcal{G}\left(\Gamma/\Gamma(N), \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \Gamma(N) \right) \right)_{N \geq 1}$$

forms a family of expanders.

Proof: Follows from the thm. on the left and Selberg's 3/16 - theorem. ∎

The building block behind this theorem is covering theory and the study of the action of

4. what distinguishes the examples from today

$$\bar{\Gamma}/\bar{\Gamma}(N) \cong PSL_2(\mathbb{Z}_N) \qquad \begin{array}{c} X_N \\ \downarrow \\ X_1 = SL_2\mathbb{Z}\backslash\mathbb{H} \end{array}$$

from the examples from last week

$$\bar{\Gamma}/\bar{\Gamma}_{\Theta} \cong \mathbb{Z}_m \qquad \begin{array}{c} \Gamma_{\Theta}\backslash\mathbb{H} \\ \downarrow \\ \Gamma\backslash\mathbb{H} \end{array} \qquad ?$$

the Galois group

$$\bar{\Gamma}/\bar{\Gamma}_i \curvearrowright \begin{array}{c} \Gamma_i\backslash\mathbb{H} \\ \downarrow \\ \Gamma\backslash\mathbb{H} \end{array} \quad \text{(by isometries)}$$

A difference between the two finite groups $\mathbb{Z}_m$ and $PSL_2(\mathbb{Z}_N)$ is well illustrated in the language of representation theory

Any nontrivial linear representation of $PSL_2(\mathbb{Z}_p)$ has dimension $\geq \frac{p-1}{2}$

while any nontrivial rep. of $\mathbb{Z}_m$ is one-dimensional

This is what motivates the assertion of Brooks.