Today:

- Isometries and geodesics in Riemannian geometry
- Spectral theory of $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

(M, g) Riemannian mfld

$\gamma : [a,b] \longrightarrow M$ piecewise C^1

has length

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ inner product

Example: $M = \mathbb{R}^n$, $p \in \mathbb{R}^n$

$$T_p\mathbb{R}^n = \left\{ \gamma'(t) : \begin{matrix} \gamma \text{ curve} \\ \gamma(t) = p \end{matrix} \right\}$$

$$\cong \mathbb{R}^n$$

std. Euclidean metric

$g_p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \; (u,v) \mapsto u \cdot v$

$$\Rightarrow L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$$

Fact:

$d : M \times M \longrightarrow \mathbb{R}$

$$d(p,q) = \inf_{\gamma} L(\gamma)$$

(over all curves that join p to q)

is a distance function.

Def: (X, d_X), (Y, d_Y) are metric spaces

A bijection $f : X \to Y$ is a (metric) isometry if

$$d_Y(f(x), f(y)) = d_X(x,y)$$

$\forall x, y \in X$.

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

<u>Def</u>: (M, g), (N, h) Riem. mflds.
A diffeomorphism $f : M \to N$ is a (Riemannian) isometry if $\forall$ curves γ in M, $L(f \circ \gamma) = L(\gamma)$

So Riemannian isometries are metric isometries for the associated distance function.

<u>Note</u>: $L(f\circ\gamma) = L(\gamma)$

$$\Longleftrightarrow \int \sqrt{h_{f(\gamma(t))}(Df \cdot \gamma'(t), Df \cdot \gamma'(t))}\, dt$$

$$= \int \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$$

<u>Geodesics</u>

<u>Def</u>: (X, d) metric space, γ curve on X. Its image is a (metric) geodesic if $\exists\, \lambda > 0$ s.t.

$$d(\gamma(t+\varepsilon), \gamma(t)) = \lambda \cdot \varepsilon$$

$\forall \varepsilon > 0$ small.

<u>Def</u>: (M, g) Riemannian mfld. Let $\gamma : [a, b] \to M$ piecewise C^1 curve paremetrized proportionally to arclength. Its image is a geodesic on M if

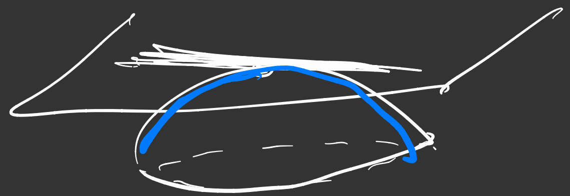
$$d(\gamma(t), \gamma(t+\varepsilon)) = L(\gamma|_{[t, t+\varepsilon]})$$

$\forall \varepsilon > 0$ small.

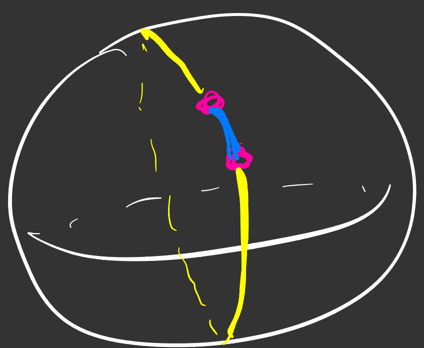
Examples:

- $\mathbb{R}^n$ geodesics = straight line segments

- S^n $T_p S^n \cong \mathbb{R}^n$
Equip S^n with std. Euclidean metric.



geodesics are arcs of great circles



geodesics are locally distance minimizing

Rmk ① Isometries preserve geodesics.

② $\mathrm{Isom}(M,g) = \{ f : M \to M \text{ isometries} \}$ is a group

Fact: (M,g), (N,h) Riem. mflds.
A diffeomorphism $f : M \to N$ is an isometry if and only if it preserves the Laplacian; $\forall \phi \in C^\infty(N)$

$$\Delta_g(\phi \circ f) = \Delta_h \phi \circ f.$$

In $\mathbb{R}^n$, $\Delta = \mathrm{div} \circ \mathrm{grad} = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$

In $(M,g) \rightsquigarrow \Delta_g$

Examples:

$$\mathrm{Isom}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n$$

$$\mathrm{Isom}(\mathbb{S}^n) \cong O(n+1)$$

<u>Prop</u>: Each isometry $f \in \mathrm{Isom}(\mathbb{R}^n)$ is of the form

$$f(x) = Ax + b,$$

where $A \in O(n)$, $b \in \mathbb{R}^n$

<u>Proof</u>:

$f \in \mathrm{Isom}(\mathbb{R}^n) \Rightarrow$ f preserves the dot prod.

f preserves lines

If $f \in \mathrm{Isom}(\mathbb{R}^n)$,

$$f \in O(n) \iff f(0) = 0.$$

In other words, each isometry is up to a translation, a linear orthogonal transfo. $\square$

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \{ x + \mathbb{Z}^n : x \in \mathbb{R}^n \}$$
$$\simeq [0,1)^n$$

- Define metric on $\mathbb{T}^n$ so that

$$\pi : \mathbb{R}^n \to \mathbb{T}^n \quad \text{canonical projection}$$

is a local isometry

- geodesics on $\mathbb{T}^n$ are projections of lines in $\mathbb{R}^n$
- $$d(x + \mathbb{Z}^n, y + \mathbb{Z}^n) = \min_{\xi \in \mathbb{Z}^n} \| x - y + \xi \|$$

$\rightsquigarrow$ $(\mathbb{T}^n, d)$ as a compact metric space

- $x \in \mathbb{R}^n \mapsto x + \xi$, $\xi \in \mathbb{Z}^n$, are isometries of $\mathbb{R}^n$

$\Rightarrow$ $\Delta = \mathrm{div} \circ \mathrm{grad}$ descends to $\mathbb{T}^n$

Solving $\Delta\varphi = \lambda\cdot\varphi$:

For $\xi \in \mathbb{Z}^n$, $\varphi_\xi(x) := e^{2\pi i\, x\cdot\xi}$

$= e(x\cdot\xi)$

This ft. descends to a ft. on $\mathbb{T}^n$

$$\Delta\varphi_\xi(x) = 4\pi^2\|\xi\|^2\varphi_\xi(x)$$

Rmk: Convention is to use the geometric Laplacian

$$\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right)$$

$L^2(\mathbb{T}^n)$ Hilbert space wrt. the following inner product

$$\langle f, g\rangle = \int_{\mathbb{T}^n} f(x)\,\overline{g(x)}\,dx$$

Prop: The $\{\varphi_\xi\}_{\xi\in\mathbb{Z}^n}$ is an orthonormal family in $L^2(\mathbb{T}^n)$.

Proof: Let $\xi, \eta \in \mathbb{Z}^n$,

$$\langle\varphi_\xi, \varphi_\eta\rangle = \int_{\mathbb{T}^n}\varphi_\xi(x)\,\overline{\varphi_\eta(x)}\,dx$$

$$= \prod_{i=1}^n \underbrace{\int_0^1 e(x_i(\xi_i - \eta_i))\,dx_i}_{= \begin{cases} 1 & \text{if } \xi_i = \eta_i \\ 0 & \xi_i \neq \eta_i \end{cases}}$$

$$= \begin{cases} 1 & \xi = \eta \\ 0 & \text{if not} \end{cases}$$

$\boxtimes$

Exercise: Use the (complex version of) Stone–Weierstrass to show that $\{\varphi_\xi\}_{\xi\in\mathbb{Z}^n}$ is dense in $L^2(\mathbb{T}^n)$.

For $\xi \in \mathbb{Z}^n$, $\varphi_\xi(x) := e^{2\pi i\, x\cdot\xi}$
$= e(x\cdot\xi)$

$\{\varphi_\xi\}_{\xi\in\mathbb{Z}^n}$ ON family

Fourier analysis:

$\forall f \in C^\infty(\mathbb{T}^n)$

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} \hat{f}(\xi)\cdot\varphi_\xi(x)$$

$$\hat{f}(\xi) = \langle f, \varphi_\xi\rangle = \int_{\mathbb{T}^n} f(x)\,\overline{\varphi_\xi(x)}\,dx$$

Connections with arithmetic $\boxed{\mathbb{T}^2}$

Each distinct eigenvalue of Δ is of the form $4\pi^2 n$, $n\in\mathbb{N}$,

$$\Delta\varphi_\xi(x) = 4\pi^2\|\xi\|^2\varphi_\xi(x)$$

and with multiplicity

$$r_2(n) := \#\{\xi\in\mathbb{Z}^2 : \|\xi\|^2 = n\}$$
$$= \#\{(a,b)\in\mathbb{Z} : n = a^2+b^2\}$$

Aside:

Every odd # is either of the form $4k+1$ or $4k+3$

$p \equiv 1\ (4)$	$3\ (4)$
$5 = 2^2+1^2$	3
$13 = 3^2+2^2$	7
$17 = 4^2+1^2$	11
$29 = 5^2+2^2$	19
	23
⋮	⋮

Fermat: Any $p\equiv 1\ (4)$ can be written as a sum of two squares.

Lagrange: Any $n \geq 1$ can be

written as the sum of 4 squares

Gauss:

$$r_2(n) := \#\{\xi \in \mathbb{Z}^2 : \|\xi\|^2 = n\} = 4 \sum_{\substack{d \geq 1 \\ d \mid n}} \chi(d)$$

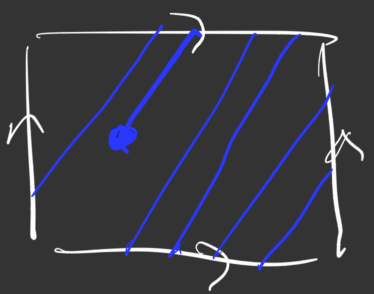
$$\chi(d) = \begin{cases} 1 & d \equiv 1\ (4) \\ -1 & d \equiv 3\ (4) \\ 0 & \text{otherwise} \end{cases}$$

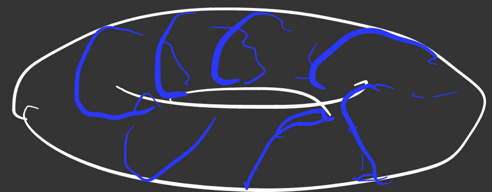
A consequence is that

$$r_2(n) \leq 4 \underbrace{\#\{d \geq 1 : d \mid n\}}_{\ll\, n^{\varepsilon} \quad (\forall \varepsilon > 0)}$$

Geodesics on $\mathbb{T}^2$

$$t \mapsto x + ty + \mathbb{Z}^n \qquad \text{for fixed } x, y \in \mathbb{R}^n$$

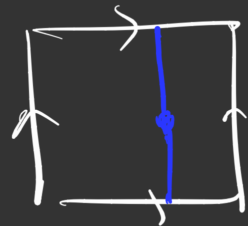


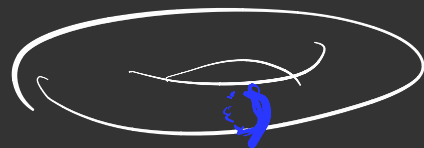


Thm: Each geodesic on $\mathbb{T}^2$ is either periodic or it becomes equidistributed in $\mathbb{T}^2$, i.e., $\forall f \in C^\infty(\mathbb{T}^2)$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x + ty)\, dt = \int_{\mathbb{T}^2} f(x)\, dx$$

(⊛)





Proof:

1. slope of $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{y_2}{y_1} = \frac{p}{q} \in \mathbb{Q}$

$p, q \in \mathbb{Z}$, $q > 0$, $(p,q) = 1$

$$ty = ty_1 \begin{pmatrix} 1 \\ p/q \end{pmatrix} \quad \text{choose } t = q/y_1$$

$$= \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{Z}^2$$

$$x + ty \equiv x \pmod{\mathbb{Z}^2}$$

(if $y_1 = 0$, then consider $\circledast$)

2. slope of $y \notin \mathbb{Q}$

We want to show $\forall f \in C^\infty(\mathbb{T}^2)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x+ty)\, dt = \int_{\mathbb{T}^2} f(x)\, dx$$

$$\int_0^T f(x+ty)\, dt$$

$\circledast$

$$= \sum_{\xi \in \mathbb{Z}^2} \hat{f}(\xi) \varphi_\xi(x) \int_0^T \varphi_\xi(ty)\, dt$$

$$= \hat{f}(0) T + \sum_{\xi \neq 0} \hat{f}(\xi) \varphi_\xi(x) \frac{e(Ty \cdot \xi) - 1}{2\pi i \xi \cdot y}$$

Rmk: $\xi \cdot y = 0$ iff $\xi = 0$

(since y has irrational slope)

$$\hat{f}(0) = \int_{\mathbb{T}^n} f(x)\, dx$$

lemma: If $f \in C^\infty(\mathbb{T}^n)$,

then $\forall k \geq 1$ $\exists C_{k,f} > 0$

s.t.

$$|\hat{f}(\xi)| \leq C_{k,f} \|\xi\|_\infty^{-k}$$

Bounding the sum:

$$|\cdots| \leq \sum_{\xi \neq 0} |\hat{f}(\xi)| \frac{2}{2\pi |\xi \cdot y|} < \infty$$

by above lemma.

Divide both sides of ⊛ by T and let $T \to \infty$

$\square$

Proof (exercise)

$\hat{f}(\xi) = \int_{\mathbb{T}^n} f(x) \overline{\psi_\xi(x)}\, dx$ and integrate by parts.