$$\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$$

<u>spectrum</u> $\{\lambda_k = 4\pi^2 k : k \in \mathbb{N}\}$

each eigenvalue appearing with multiplicity $r_n(k) = \#$ repr. of k as a sum of n squares

<u>spectral decomposition</u>

$L^2(\mathbb{T}^n)$ admits an ONB $\{\varphi_\xi\}_{\xi \in \mathbb{Z}^n}$ of Δ-eigenfunctions.

In particular, for each $f \in C^\infty(\mathbb{T}^n)$

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \langle f, \varphi_\xi \rangle \varphi_\xi(x).$$

$$\varphi_\xi(x) = e(\xi \cdot x) \qquad x \in \mathbb{T}^n$$

<u>Notation</u> $e(x) := e^{2\pi i x}$

<u>Rmk</u>: $\Delta f = \lambda f$

(*) $\lambda \|f\|^2 = \|\nabla f\|^2 \rightsquigarrow \lambda \geq 0$

Today:

- Examine spectral situation $L^2(\mathbb{R}^n)$
- Poisson summation formula

<u>Prop</u>: $\mathcal{D} = \{f \in L^2(\mathbb{R}^n) : \Delta f \in L^2(\mathbb{R}^n)\}$

The geometric Laplacian Δ is a linear, selfadjoint, nonnegative operator on $\mathcal{D}$.

<u>Proof</u>:

Recall $L^2(\mathbb{R}^n)$ is equipped with the std. inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)}\, dx$$

$$\langle \Delta f, g \rangle = \int_{\mathbb{R}^n} \nabla f(x) \overline{\nabla g(x)}\, dx = \langle f, \Delta g \rangle$$

$$\langle \Delta f, f \rangle = \|\nabla f\|_{L^2}^2 \geq 0 \qquad (*)$$

$\square$

Thm (Friedrich)
If a linear operator T is selfadj. and nonnegative on a dense subspace of a Hilbert space $\mathcal{H}$ then T admits a selfadjoint extension to $\mathcal{H}$.

N.B. $C_c^\infty(\mathbb{R}^n) \subset \mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$ hence Δ extends to a linear self adjoint operator on $L^2(\mathbb{R}^n)$.

On $L^2(\mathbb{R}^n)$,

$u \in \mathbb{R}^n$ $\quad \varphi_u(x) = e(u \cdot x)$

$$\Delta \varphi_u = 4\pi^2 \|u\|^2 \varphi_u$$

$\rightsquigarrow$ The spectrum of Δ on $\mathbb{R}^n$ is as large possible, $[0, \infty)$

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ $\quad D^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\varphi_u \notin L^2$ $\qquad \int_{\mathbb{R}^n} \underbrace{|\varphi_u|^2}_{=1} = \infty$

Fourier analysis

If $f \in L^1(\mathbb{R}^n)$, then its Fourier transform

$$\hat{f}(u) = \int_{\mathbb{R}^n} f(x) \overline{\varphi_u(x)}\, dx,$$

and if $\hat{f} \in L^1(\mathbb{R}^n)$,

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(u) \varphi_u(x)\, du$$

def: A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is Schwartz if for any $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty$$

exercise: $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ ← Schwartz space of functions

Thm: (Poisson summation formula)

If $f \in \mathcal{S}(\mathbb{R}^n)$ then

$$\sum_{\xi \in \mathbb{Z}^n} f(\xi) = \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi)$$

Proof:

Define $F : \mathbb{R}^n \to \mathbb{C}$

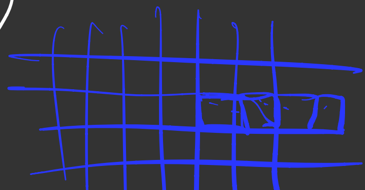
$$F(x) = \sum_{\xi \in \mathbb{Z}^n} f(x+\xi)$$

is well defined since $f \in \mathcal{S}(\mathbb{R}^n)$.

Moreover, $\forall \eta \in \mathbb{Z}^n$

$$F(x+\eta) = \sum_{\xi \in \mathbb{Z}^n} f(x+\xi+\eta) = F(x)$$

and $F \in C^\infty(\mathbb{T}^n)$



Thus F admits the expansion

$$F(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{F}(\xi) \varphi_\xi(x)$$

"Taking the trace":

$$\sum_{\xi \in \mathbb{Z}^n} f(\xi) = \sum_{\xi \in \mathbb{Z}} \hat{F}(\xi)$$

$$\hat{F}(\xi) = \int_{\mathbb{T}^n} F(x) \overline{\varphi_\xi(x)}\, dx$$

$$= \sum_{\eta \in \mathbb{Z}^n} \int_{\mathbb{T}^n} f(x+\eta) \overline{\varphi_\xi(x)}\, dx$$

$x+\eta \to x$

$$= \sum_{\eta \in \mathbb{Z}^n} \int_{[0,1)^n+\eta} f(x) \overline{\varphi_\xi(x)}\, dx$$

$$= \int_{\mathbb{R}^n} f(x) \overline{\varphi_\xi(x)}\, dx = \hat{f}(\xi)$$

Suppose we have an ONB $\{\varphi_k\}_{k\geq 0}$, $\Delta\varphi_k = \lambda_k\varphi_k$ with

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$$

$$Z(t) = \sum_{k\geq 0} e^{-t\lambda_k} \qquad (t>0)$$

is called the <u>spectral partition function</u> and it completely determines the spectrum of Δ

Compare to moment-generating ft: X random variable

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Replace $\{\lambda_k\}$ by $\{\tilde{\lambda}_k\}$ where

$$0 \leq \tilde{\lambda}_0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$$

each $\tilde{\lambda}_k$ with multiplicity μ_k

$$Z(t) = \sum_{k\geq 0} \mu_k e^{-t\tilde{\lambda}_k}$$

$$\lim_{t\to\infty}\left[e^{rt} Z(t) = \sum_{k\geq 0} \mu_k e^{(r-\tilde{\lambda}_k)t}\right]$$

$$= \begin{cases} \infty & r > \tilde{\lambda}_0 \\ 0 & r < \tilde{\lambda}_0 \\ \mu_0 & r = \tilde{\lambda}_0 \end{cases}$$

$$\lim_{t\to\infty} e^{rt}\left(Z(t) - \mu_0 e^{-\tilde{\lambda}_0 t}\right)$$

$$= \begin{cases} \infty & r > \tilde{\lambda}_1 \\ 0 & r < \tilde{\lambda}_1 \\ \mu_1 & r = \tilde{\lambda}_1 \end{cases}$$

For $\mathbb{T}^n$,

$$Z_{\mathbb{T}^n}(t) = \sum_{\xi \in \mathbb{Z}^n} e^{-4\pi^2 \|\xi\|^2 t}$$

$$= \sum_{k \geq 0} r_n(k) e^{-4\pi^2 k t}$$

is closely related to Jacobi's theta series

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z) \qquad (z \in \mathbb{H})$$

$$|\Theta(z)| \leq \sum_{n \in \mathbb{Z}} \underbrace{\left| e^{2\pi i n^2 (x+iy)} \right|}_{e^{-2\pi n^2 y}} < \infty$$

$$\Theta(z)^2 = \left(\sum_{a \in \mathbb{Z}} e(a^2 z) \right) \left(\sum_{b \in \mathbb{Z}} e(b^2 z) \right)$$

$$= \sum_{a, b \in \mathbb{Z}} e((a^2 + b^2) z)$$

$$= \sum_{k \geq 0} r_2(k) e(kz)$$

$$\Theta(z)^n = \sum_{k \geq 0} r_n(k) e(kz)$$

$$Z_{\mathbb{T}^n}(t) = \Theta(2\pi i t)^n$$

$$\tilde{\Theta}(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \qquad (t > 0)$$

<u>Thm</u>: (Jacobi's inversion formula) For any $t > 0$,

$$\tilde{\Theta}(t) = \frac{1}{\sqrt{t}} \tilde{\Theta}\left(\frac{1}{t}\right)$$

Proof: Let $f(x) = e^{-\pi x^2 t} \in \mathcal{S}(\mathbb{R})$ $(t>0)$ and apply Poisson summation

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{-2\pi i u x}\, dx$$

$$= \dots = \frac{1}{\sqrt{u}} e^{-\pi u^2/t}.$$

$$\rightsquigarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$$

$\square$

Coro: (trace formula for $\mathbb{T}^n$)

$$Z_{\mathbb{T}^n}(t) = \sum_{\xi\in\mathbb{Z}^n} e^{-4\pi^2\|\xi\|^2 t}$$

$$\overset{(*)}{=} \frac{1}{(4\pi t)^{n/2}} \sum_{\xi\in\mathbb{Z}^n} e^{-\|\xi\|^2/4t}$$

Proof:

$$Z_{\mathbb{T}^n}(t) = \Theta(2\pi i t)^n = \tilde{\Theta}(4\pi t)^n$$

$$= \frac{1}{(4\pi t)^{n/2}} \tilde{\Theta}\left(\frac{1}{4\pi t}\right)^n$$

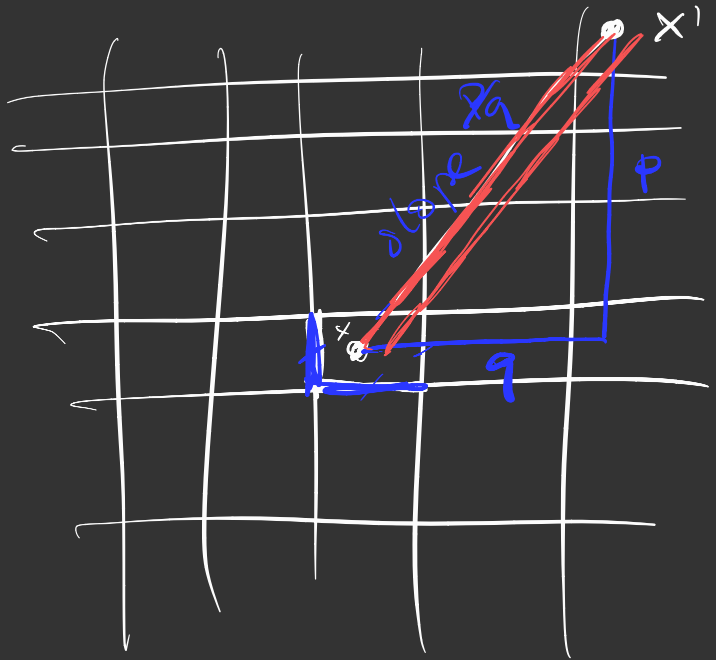
$\square$

LHS of $(*)$ is called the spectral side

RHS of $(*)$ is called the geometric side

$$\frac{1}{(4\pi t)^{n/2}} \sum_{\xi\in\mathbb{Z}^n} e^{-\|\xi\|^2/4t}$$

is a sum over all closed geodesics up to free homotopy.



$\mathbb{R}^2/\mathbb{Z}^2$

$x' \equiv x \pmod{\mathbb{Z}^2}$

this also shows that the length of a closed geodesic is of the form

$\|\xi\|$, $\xi \in \mathbb{Z}^2$

Geometrically, we're only interested in closed geodesics up to free homotopy.