Today: geometry of $\mathbb{H}$

$$\mathbb{H} = \{ x+iy \in \mathbb{C} : y > 0 \}$$

$$T_z \mathbb{H} \cong \mathbb{C}$$

$$g_z(u,v) = \frac{u\bar{v}}{y^2} \qquad y = \operatorname{Im}(z)$$

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$$

$$f: \mathbb{H} \longrightarrow \mathbb{D}$$

$$z \longmapsto \frac{z-i}{z+i}$$

the Cayley transform

biholomorphic map

$$F = f^{-1}$$

For each $z \in \mathbb{D}$,

$$h_z(u,v) = g_{F(z)}(D_zF \cdot u, D_zF \cdot v)$$

$$= \frac{|D_zF|^2 u \cdot \bar{v}}{\operatorname{Im} F(z)^2} = \frac{4 u\bar{v}}{(1-|z|^2)^2}$$

The Cayley transform is an isometry from $(\mathbb{H}, g)$ to $(\mathbb{D}, h)$.

For various computations, it is more convenient to work in the disk model.

We now determine what are

- length
- angle
- area
- curvature, Laplacian
- isometries
- geodesics

wrt the hyperbolic metric

Length

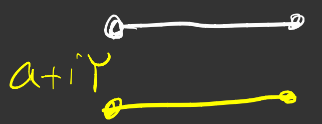
$\gamma : [a,b] \longrightarrow \mathbb{H}$ curve

$$L(\gamma) = \int_a^b \frac{|\gamma'(t)|}{\operatorname{Im}(\gamma(t))}\, dt$$

$\gamma(t) = x(t) + i y(t)$

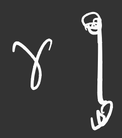
$$= \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

$\mathbb{H}$



$\gamma(t) = t + iy$

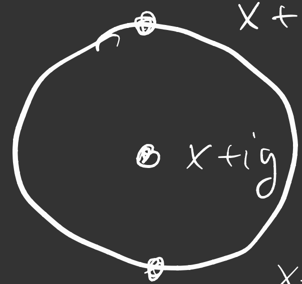
$$L(\gamma) = \frac{b-a}{y} \xrightarrow[y \to 0]{} \infty$$



$\gamma(t) = x + it$

$$L(\gamma) = \int_a^b \frac{dt}{t}$$

$$= \log \frac{b}{a}$$



Euclidean center of this circle is $x + i\,\dfrac{b-a}{2}$

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hyperbolic center is at $x + i\sqrt{ab}$

$$\log \frac{b}{y} = \log \frac{y}{a}$$

$$y = \sqrt{ab}$$

Angles:

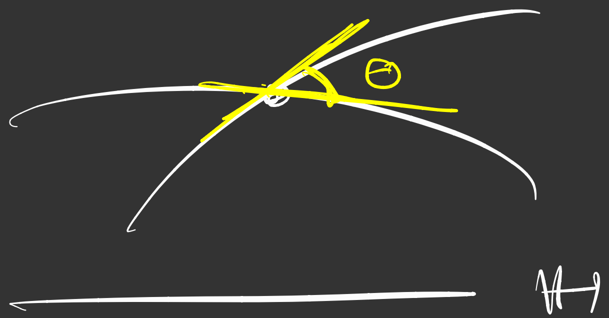
The angle θ between $u, v \in T_z\mathbb{H} \cong \mathbb{C}$ is given

$$\cos\theta = \frac{g_z(u,v)}{|u|_z |v|_z} = \frac{u\bar{v}}{y^2 \frac{|u|}{y} \cdot \frac{|v|}{y}}$$

$$|u|_z = \sqrt{g_z(u,u)}$$

$$= \frac{u \cdot \bar{v}}{|u||v|}$$

$\rightsquigarrow$ hyperbolic angles coincide with Euclidean angles



Curvature (Gaussian) and Laplacian

$K = -1$

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad \text{(geometric)}$$

Area

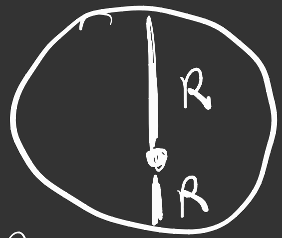
$\Omega \subset \mathbb{H}$

$$\text{area}(\Omega) = \iint_\Omega \frac{dx\,dy}{y^2}$$

$\Omega \subset \mathbb{D}$

$$\text{area}(\Omega) = \iint_\Omega \frac{4\,r\,dr\,d\theta}{(1-r^2)^2}$$

Exercise: Let D_R be a hyperbolic disk of (hyperbolic) radius R in $\mathbb{H}$.



Check that

$$\text{area}(D_R) = 2\pi \sinh^2\left(\frac{R}{2}\right) \sim e^R$$

$$\text{circ}(D_R) = 2\pi \sinh(R) \sim e^R$$

Isometries

Poincaré: $\text{Isom}^+(\mathbb{H}) \cong PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm I\}$

def: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

A Möbius transformation is a fct. $f: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$

$$z \longmapsto \frac{az+b}{cz+d}$$

$a,b,c,d \in \mathbb{C} \qquad ad-bc \neq 0$

$$\infty \longmapsto \frac{a}{c}$$

$$-\frac{d}{c} \longmapsto \infty$$

The set of all such Möbius transfo. forms a group and is isomorphic to $PGL_2\mathbb{C} = GL_2\mathbb{C}/\mathbb{C}^\times$

Prop: Möbius transformations preserve generalized circles.

any circle or line passing through ∞.

Proof: Any Möbius transfo. can be decomposed as a composition of translations $z \longmapsto z+\alpha$, $\alpha \in \mathbb{C}$

dilations $z \longmapsto \alpha \cdot z$

inversions $z \longmapsto \dfrac{1}{z}$

$$z \mapsto \frac{az+b}{cz+d}$$

$$z \mapsto cz+d \mapsto \frac{1}{cz+d} + t$$

$$= \frac{ctz + dt + 1}{cz+d}$$

$$\mapsto \frac{az + \left(\frac{ad}{c} + \frac{a}{ct}\right)}{cz+d} = \frac{az+b}{cz+d}$$

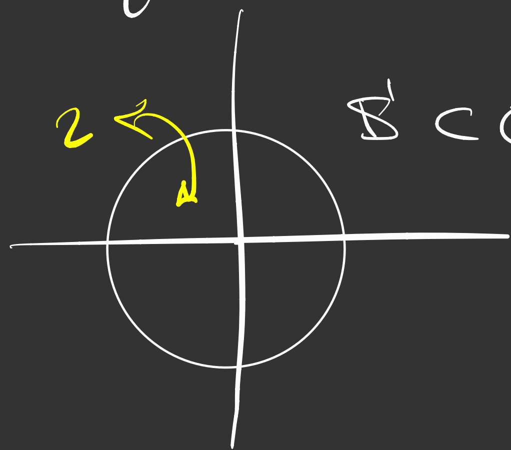
Pick $t = \dfrac{-a}{ad-bc}$

Translations and dilations preserve circles.

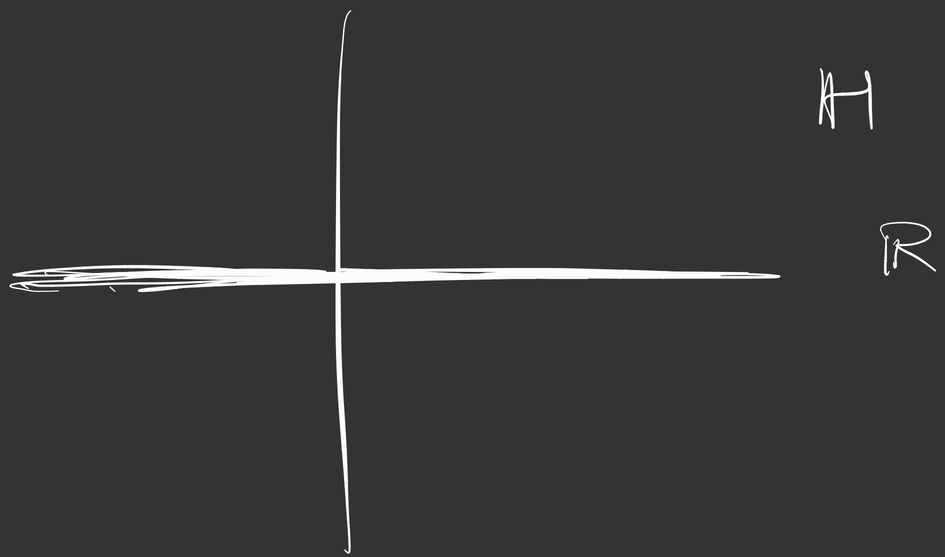
Any inversion can be written as the composition of

2: $z \mapsto \dfrac{1}{\bar{z}}$ and $z \mapsto \bar{z}$

preserve gen-circles



$S^1 \subset \mathbb{C}$ This is a circle inversion and preserves gen. circles as well. $\square$



The only Möbius transfo. in $PGL_2\mathbb{C}$ that preserve $\mathbb{R}$ are in $PGL_2\mathbb{R}$

Let $g \in PGL_2\mathbb{R}$, $z \in \mathbb{H}$

$$\operatorname{Im}(g(z)) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}(ad-bc) > 0$$

iff $ad-bc > 0$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We restrict to

$$PGL_2^+(\mathbb{R}) \cong PSL_2\mathbb{R}$$

$$g \longmapsto \frac{g}{\sqrt{\det g}}$$

In particular:

$$PSL_2\mathbb{R}(\mathbb{H}) = \mathbb{H}$$

<u>Prop</u>: $PSL_2(\mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$

$$PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm I\}$$

$$z \mapsto \frac{az+b}{cz+d} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$ad-bc=1$

$a, b, c, d \in \mathbb{R}$

Proof: An isometry is a fct. bij. $f: \mathbb{H} \to \mathbb{H}$ that preserves the length of curves.

$$\gamma : [0,1] \longrightarrow \mathbb{H}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$$

$$L(g \circ \gamma) = \int_0^1 \frac{|(g\circ\gamma)'(t)|}{\operatorname{Im}((g\circ\gamma)(t))} \, dt$$

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}$$

$$g'(z) = \frac{d}{dz} \frac{az+b}{cz+d} = \frac{1}{(cz+d)^2}$$

$$= \int_0^1 \frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)} \, dt = L(\gamma)$$

□

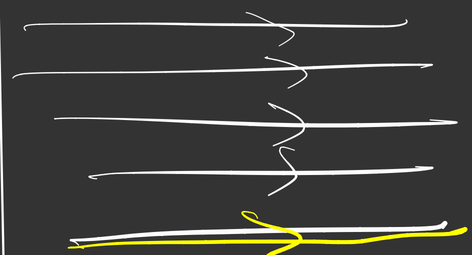
$PSL_2\mathbb{R}$ contains

translations $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : z \longmapsto z + t$ $(t \in \mathbb{R})$

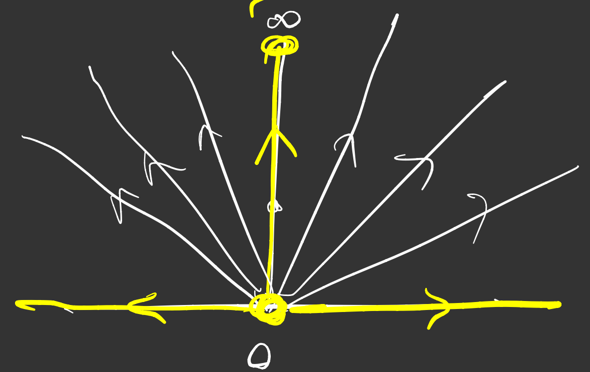
dilations $\begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix} : z \longmapsto t \cdot z$ $\quad t \in \mathbb{R}_{>0}$

involution: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \longmapsto \frac{-1}{z}$

∞ fixed by $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$



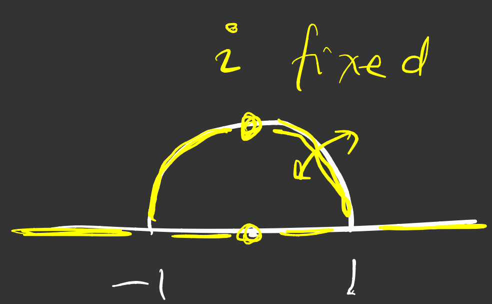
$n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \infty = \frac{1}{0} = \infty$



0 and ∞ are fixed by $a_t = \begin{pmatrix} \sqrt{t} & \\ & 1/\sqrt{t} \end{pmatrix}$

Prop:

1. $PSL_2\mathbb{R} \curvearrowright \mathbb{H}$ transitively, i.e. $\forall z \in \mathbb{H}\ \exists g \in PSL_2\mathbb{R}$
$$g(i) = z$$



$$k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

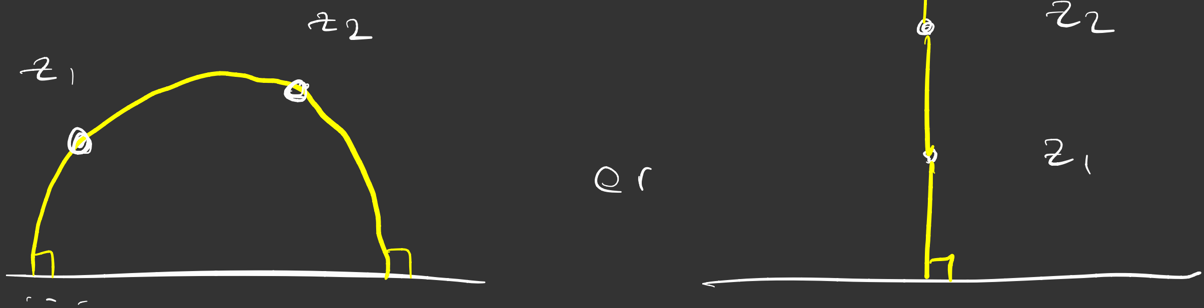
2. $PSL_2\mathbb{R}$ acts transitively on triples of points in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, i.e. $\forall (x_1, x_2, x_3) \in \hat{\mathbb{R}}^3$ $\exists g \in PSL_2\mathbb{R}$
$$(g(0), g(1), g(\infty)) = g(0, 1, \infty) = (x_1, x_2, x_3)$$

Proof:

1. $z = x + iy \qquad g = n_x a_y \qquad g(i) = n_x a_y(i) = n_x(iy) = x + iy$
2. exercise.

Geodesics

Prop: Geodesics are of the form ...



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Proof:

let $z_2 = x + ib$, $z_1 = x + ia$

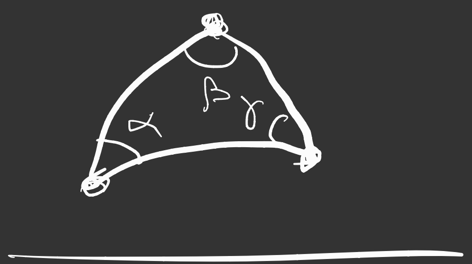
this vertical segment has hyp length $\log \frac{b}{a}$

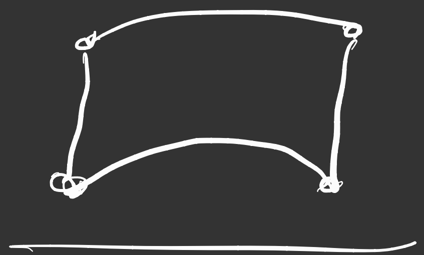
Let γ be a curve joining z_1 to z_2, $\gamma(t) = x(t) + i y(t)$

$$L(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

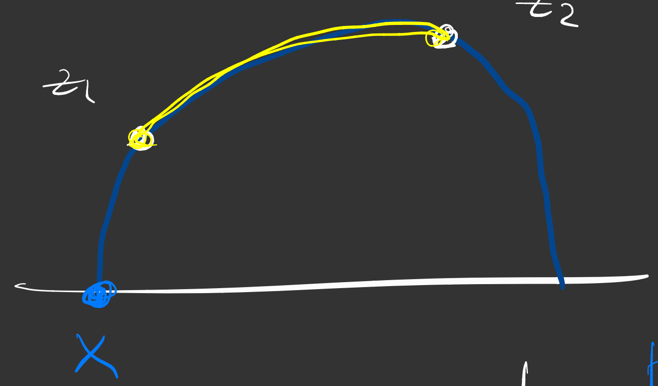
$$\geq \int_0^1 \frac{|y'(t)|}{y(t)}\, dt.$$

$$= \int_a^b \frac{dy}{y} = \log \frac{b}{a}$$



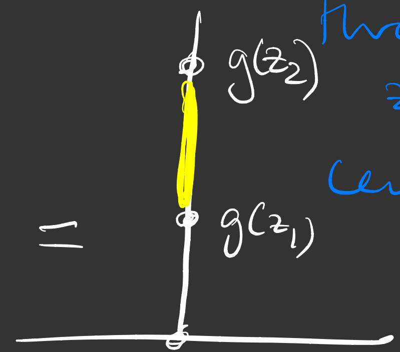


The non-aligned case



L is the semi circle passing through z_1, z_2 with center on $\mathbb{R}$

$\exists\, g \in PSL_2\mathbb{R}$ s.t. $g(L) =$



(using that Möbius transfo. preserve generalized circles)

The shortest path from $g(z_1)$ to $g(z_2)$ is the vertical segment connecting them.

Since $g \in \mathrm{Isom}(\mathbb{H})$, the shortest path btw. z_1 and z_2 is along L. $\square$

Prop: Let T be a hyperbolic triangle (as above). Its area is

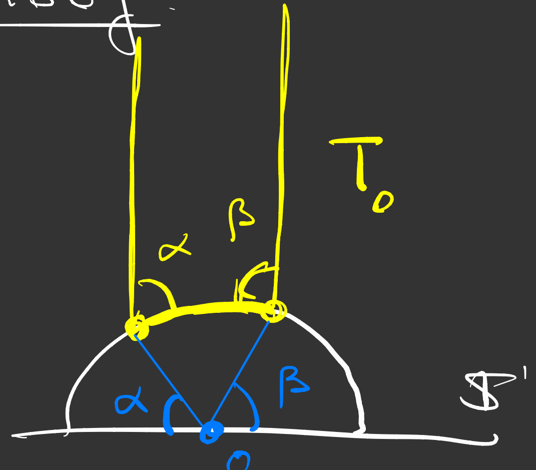
$$\text{area}(T) = \pi - \alpha - \beta - \gamma .$$

Rmk: The sum of the inner angles of T is thus $< \pi$

Rmk: This is a special case of the Gauss-Bonnet theorem.

Proof: ∞

here the angle is 0

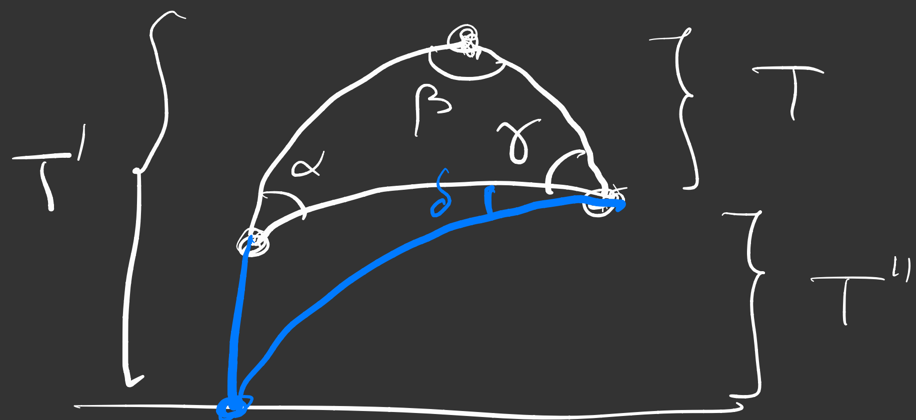


$$\text{area}(T_0) = \int_{\cos(\pi-\alpha)}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2}$$

$$= \int_{\sin(-\frac{\pi}{2}+\alpha)}^{\sin(\frac{\pi}{2}-\beta)} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} - \beta + \frac{\pi}{2} - \alpha$$

$$= \pi - \alpha - \beta$$

Looking at a hyperbolic triangle in general position



$$\text{area}(T) = \text{area}(T') - \text{area}(T'')$$

Up to an isometry, every triangle with a vertex on $\hat{\mathbb{R}}$ can be brought into the form T_0 (*)

$$= \pi - \beta - (\gamma + \delta) - (\pi - (\pi - \alpha) - \delta)$$

$$= \pi - \beta - \alpha - \gamma$$

(*) Use that $PSL_2\mathbb{R}$ acts transitively on triples of points in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Next time:

- metric properties of $\mathbb{H}$
- classification of isometries in $PSL_2\mathbb{R}$