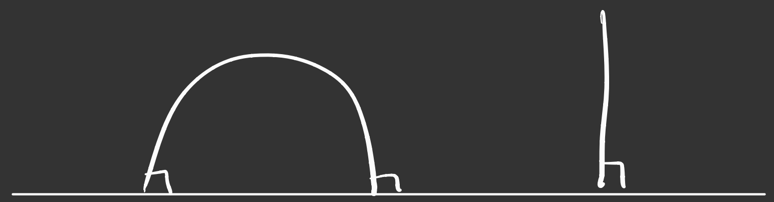
Recap

$\mathbb{H}$ with $g_z(u,v) = \frac{u \cdot \bar{v}}{y^2}$ $\left(\leftrightarrow ds^2 = \frac{dx^2 + dy^2}{y^2} \right)$

geodesics are portions of



$PSL_2\mathbb{R} \subset \mathrm{Isom}(\mathbb{H})$

(In fact:

Thm (Poincaré) $\mathrm{Isom}^+ \mathbb{H} = PSL_2\mathbb{R}$)

Today:

- Classification of motions

Recall: $\mathrm{Isom}^+ \mathbb{R}^2 = SO(2) \ltimes \mathbb{R}^2$

Any $\varphi \in$ is either a rotation or a translation or the identity

- Metric space properties of $(\mathbb{H}, ds^2)$

- Fuchsian groups

Classification of motions

$G = PSL_2\mathbb{R}$ contains

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} / \{\pm 1\}$$

$z \mapsto z + x$

$$A = \left\{ a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} : y \in \mathbb{R}_{>0} \right\} / \{\pm 1\}$$

$z \mapsto y \cdot z$

Recall: $NA(i) = \mathbb{H}$

$$K = \left\{ k_\theta = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Check: $\mathrm{Stab}_G(i) = \{ g \in G : g(i) = i \}$
$= K$

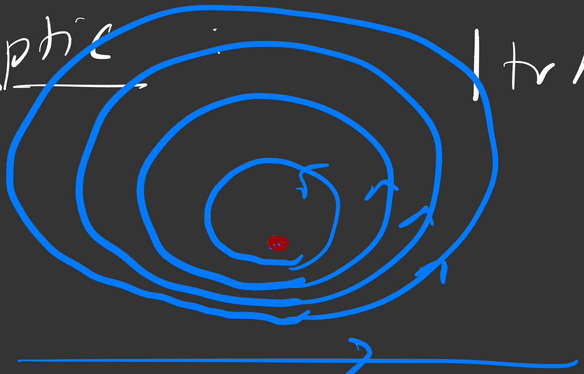
Thm: Each $g \in G$, $g \neq \pm I$, is either ...

$g = \pm A$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$

Rmk: Iwasawa decomposition of G: Each $g \in G$ can be written (uniquely) as $g = nak$, $n \in N$, $a \in A$, $k \in K$

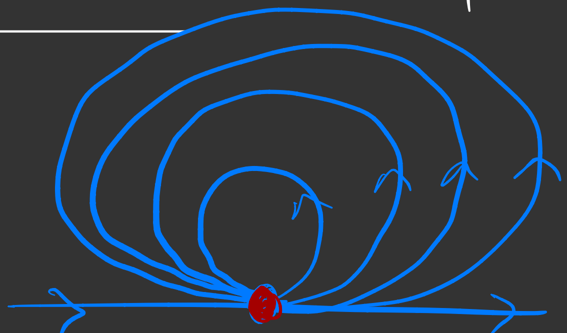
$\overline{\mathbb{H}} = \mathbb{H} \cup \underbrace{\partial\mathbb{H}}_{\mathbb{R} \cup \{\infty\}}$

elliptic: $|\mathrm{tr}\, A| < 2 \longleftrightarrow$ $g \curvearrowright \overline{\mathbb{H}}$ has a unique fixed pt. & it is in $\mathbb{H}$ $\longleftrightarrow$ g conjugate to some $k \in K$



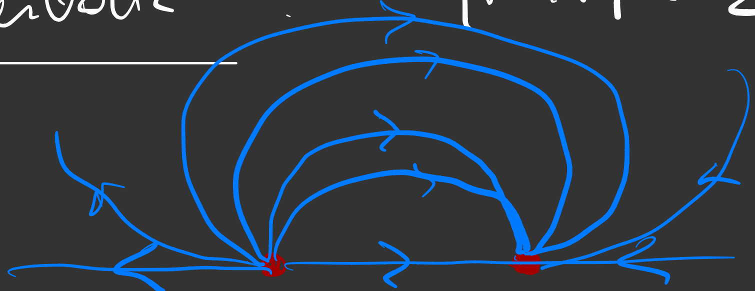
$\mathbb{H}$

parabolic: $|\mathrm{tr}\, A| = 2 \longleftrightarrow$ $g \curvearrowright \overline{\mathbb{H}}$ has a unique fixed pt & it is in $\partial\mathbb{H}$ $\longleftrightarrow$ g conjugate to some $n \in N$



$\mathbb{H}$

hyperbolic: $|\mathrm{tr}\, A| > 2 \longleftrightarrow$ $g \curvearrowright \overline{\mathbb{H}}$ has two fixed pts (both in $\partial\mathbb{H}$) $\longleftrightarrow$ g conjugate to some $a \in A$



$\mathbb{H}$

<u>Proof</u>: $Az = z \longleftrightarrow az+b = cz^2 + dz \longleftrightarrow cz^2+(d-a)z - b = 0$

with discriminant $\Delta = (a+d)^2 - 4$

if A is elliptic $|\mathrm{tr} A| < 4 \longleftrightarrow \Delta < 0$

and $Az = z$ has solutions

$$\frac{a-d \pm i\sqrt{|\Delta|}}{2c}$$ only one is in $\mathbb{H}$

There is $g \in G$ s.t. $g(i) = z$

$Az = z \longleftrightarrow \underbrace{g^{-1}Ag}_{\in PSL_2 \mathbb{R}}(i) = i \longrightarrow g^{-1}Ag \in K$

$\boxtimes$

Metric properties of $\mathbb{H}$

For each $z_1, z_2 \in \mathbb{H}$

$$d_{\mathbb{H}}(z_1, z_2) = \inf_{\gamma} L(\gamma)$$

Some explicit formulas:

$z_1, z_2 \in \mathbb{H}$:

$$(*) \quad \cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2 y_1 y_2}$$

$z_1, z_2 \in \mathbb{D}$:

$$\tanh \tfrac{1}{2} d_{\mathbb{D}}(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - z_1 \bar{z}_2|}$$

Rmks:

1. Check that both sides of $(*)$ are invariant under $PSL_2\mathbb{R}$

Up to isometry, take $z_1 = ia$, $z_2 = ib$.

and check $(*)$.

2. Can equip $SL_2\mathbb{R}$ with the matrix topo. induced by

$$\|A\|^2 = a^2 + b^2 + c^2 + d^2$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$.

Check: $\|A\|^2 = 2\cosh d_{\mathbb{H}}(i, Ai)$

Prop: $(\mathbb{H}, d_{\mathbb{H}})$ and $(\mathbb{D}, d_{\mathbb{D}})$ are complete metric space.

Proof: It suffices to prove this for $(\mathbb{D}, d_{\mathbb{D}})$.

Let $(z_n) \subset \mathbb{D}$ Cauchy wrt $d_{\mathbb{D}}$

Then (z_n) also <u>Cauchy</u> wrt $d_{\mathbb{C}}$

$(\forall \varepsilon > 0 \ \exists N > 0$ s.t. $\forall m, n \geq N$

$|z_n - z_m| < \varepsilon$)

since

$$|z_n - z_m| \leq 2 \tanh \tfrac{1}{2} d_{\mathbb{D}}(z_m, z_n)$$

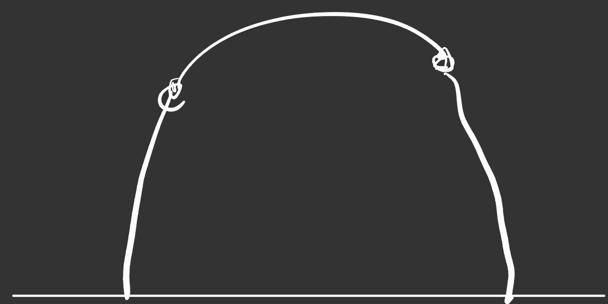
Hence $z_n \longrightarrow z \in \mathbb{D}$ wrt $d_{\mathbb{C}}$

Similarly, $z_n \to z$ wrt $d_{\mathbb{D}}$ using the above formula $\square$

<u>Thm</u> (Hopf–Rinow)

(M, g) connected Riem. mfld. Then the following are equivalent:

1. (M, d_M) is a complete metric space
2. M is geodesically complete (i.e., every geodesic can be extended infinitely)
3. Closed bounded subsets of M are compact.



Fuchsian groups

<u>def</u>: $\Gamma < SL_2\mathbb{R}$ is Fuchsian if it is discrete wrt the induced matrix topology

($\leftrightarrow$ $\forall C > 0$ $\{\gamma \in \Gamma : \|\gamma\| < C\}$ is at most finite)

Fuchsian grps. are named after Fuchs. Poincaré was studying his work on certain second order linear DE, and came to the following

<u>task</u>: Find $F : \mathbb{H} \to \mathbb{C}$ holomorphic and Γ-invariant for some $\Gamma < SL_2\mathbb{R}$ acting on $\mathbb{H}$ by Möbius transform.

Factsheet of equivalent characterizations in this context:

1. Γ is Fuchsian
2. $\Gamma \curvearrowright \mathbb{H}$ has <u>discrete orbits</u>: no orbit Γz has accumulation pt. in $\mathbb{H}$
3. $\Gamma \curvearrowright \mathbb{H}$ is <u>properly discont.</u>: $\forall K \subset \mathbb{H}$ compact
$$K \cap \gamma K = \emptyset$$
except for at most fin.-many $\gamma \in \Gamma$
4. $\Gamma \curvearrowright \mathbb{H}$ is <u>wandering</u>: $\forall z \in \mathbb{H}$ $\exists\, \mathcal{U}$ nbhd. of z s.t.
$$\mathcal{U} \cap \gamma\mathcal{U} \neq \emptyset \Rightarrow \gamma \in \mathrm{Stab}_\Gamma(z)$$
and $\mathrm{Stab}_\Gamma(z)$ is finite

⇝ restrict to actions with discrete orbits.

If Γz has an acc. pt, then there is a nbhd. containing ∞ many $w \in \Gamma z$

On each one, $f(w) = F(z)$

If F is holo. $\Rightarrow$ $F \equiv f(z)$ constant

Prop: X is a Hausdorff, loc. cpt. space, $\Gamma \curvearrowright X$ prop. discont.

Then $\Gamma \backslash X$ is Hausdorff.

If the action is moreover free, then $X \longrightarrow \Gamma\backslash X$

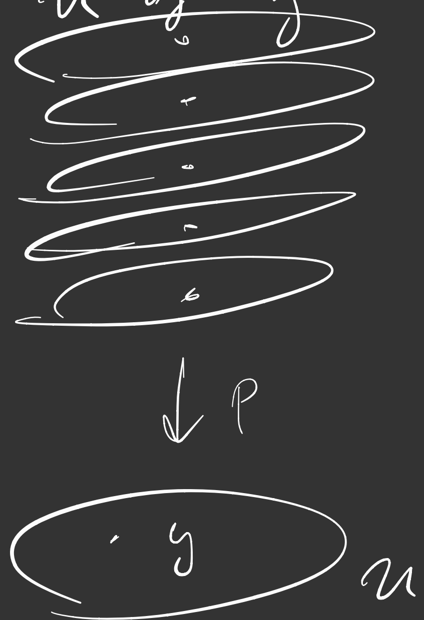
($\gamma x = x \Rightarrow \gamma = e$) is a covering projection.

covering projection.

$$\begin{array}{c} X \\ p \big\downarrow \\ Y \end{array}$$

covering projection: if p cont., surj.

and for each $y \in Y$, $\exists$ a nbhd $\mathcal{U}$ of y s.t.



$$p^{-1}(\mathcal{U}) = \coprod_\alpha V_\alpha$$

s.t.

$p(V_\alpha) \cong \mathcal{U}$ (homeo)

Proof: Let $x \in X$. Since $\Gamma \curvearrowright X$ freely & prop. disc., there is a nbhd $\mathcal{U} \ni x$ that is disjoint from any other pt. in Γx.

Hence, given the projection map $p : X \longrightarrow \Gamma \backslash X$

$$\mathcal{U} \approx p(\mathcal{U})$$

$\square$

Example:

$$\begin{array}{c} \mathbb{H} \\ \downarrow \\ \Gamma \backslash \mathbb{H} \end{array}$$

$\Gamma \curvearrowright \mathbb{H}$ prop. discont.

$\Gamma \curvearrowright \mathbb{H}$ free

$\iff$ Γ contains no elliptic motions

f is a deck transformation if $f \in \mathrm{Homeo}(X)$ and $p \circ f = p$

and the set of all deck transformations of a group. D

If X is connected, then

$D \curvearrowright X$ prop. discont. & freely

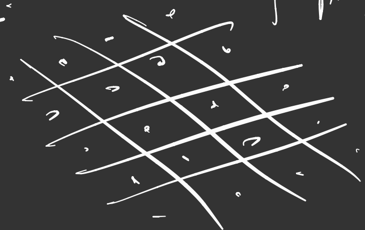
If X is simply connected, then $D \cong \pi_1(Y, y)$

and

$$Y = \pi_1(Y, y) \backslash X$$

Example:

$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ \mathbb{T}^2 \end{array}$$



prop. disc. & free

$\curvearrowleft \mathbb{Z}^2$

$\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$

$\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

Given $\Gamma \curvearrowright \mathbb{H}$ free + prop-disc, one can transport the smooth/Riemannian/hyperbolic structure of $\mathbb{H}$ to $\Gamma \backslash \mathbb{H}$ (p is a local isometry)

$M = \Gamma \backslash \mathbb{H}$ is a C^∞ surface & a hyperbolic surface.

Actually: all hyperbolic surfaces arise this way

M hyp surface admits a universal cover

$\tilde{M} = \mathbb{H}$

$\pi_1(M) \cong \Gamma < PSL_2\mathbb{R}$ which acts prop. dis. & freely on $\mathbb{H}$.

$$\begin{array}{c} \tilde{M} \\ \downarrow \\ M \end{array} = \pi_1(M) \backslash \tilde{M}$$

$\pi_1(M) \curvearrowright \tilde{M}$ by isom.

<u>Hopf</u>:

Up to isometry and "global ~~re~~scaling", the only C^∞ complete simply connected surfaces of constant curvature are

$\mathbb{R}^2$ $(K = 0)$

S^2 $(K = 1)$

$\mathbb{H}^2$ $(K = -1)$