def: $\Gamma < SL_2\mathbb{R}$ Fuchsian if discrete

Fact:

Γ Fuchsian $\iff$ $\Gamma \curvearrowright \mathbb{H}$ prop. disc.

$\iff$ all Γ-orbits are discrete

$\Gamma \curvearrowright \mathbb{H}$ prop. disc. + free

$(\gamma z = z \implies \gamma = \pm I)$

$$\mathbb{H} \xrightarrow{\text{covering (loc. isometry)}} \Gamma\backslash\mathbb{H}$$

$\Gamma\backslash\mathbb{H}$ is a hyperbolic surface

If M is a hyperbolic surface then

$$\mathbb{H} \downarrow M = \Gamma\backslash\mathbb{H}$$ where Γ Fuchsian acting prop. discont. and freely on $\mathbb{H}$.

Similar picture for

$\mathbb{Z}^2 < \mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$

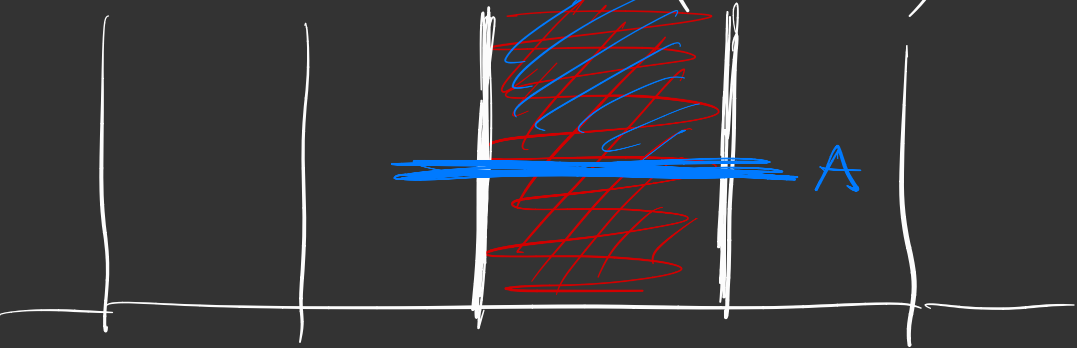
$$\mathbb{R}^2 \downarrow \mathbb{R}^2/\mathbb{Z}^2$$

$[0,1)^2$ is a fundamental domain for the $\mathbb{Z}^2$-action on $\mathbb{R}^2$

def: $\mathcal{F} \subset \mathbb{H}$ connected set is a fundamental domain for the action of a Fuchsian group Γ on $\mathbb{H}$ if

Examples $(t>0)$

1. $\Gamma = \left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & t\mathbb{Z} \\ 0 & 1 \end{pmatrix}$



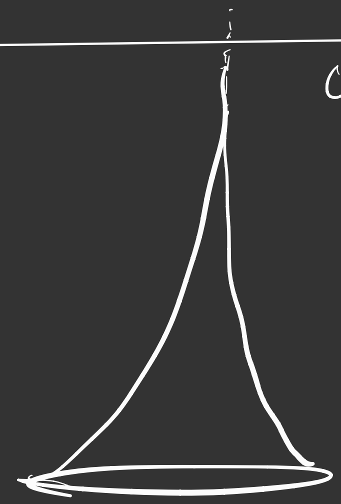
$z \mapsto z+t$

$$\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma \bar{F}$$

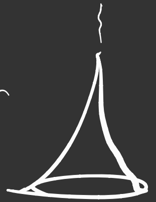
and $\gamma_1 \mathring{F} \cap \gamma_2 \mathring{F} = \emptyset$ if $\gamma_1 \neq \gamma_2$ in Γ

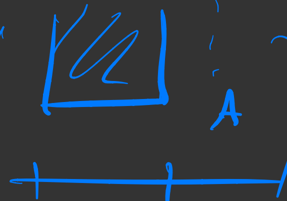
$\mathbb{H} \to \Gamma\backslash\mathbb{H}$

cusp



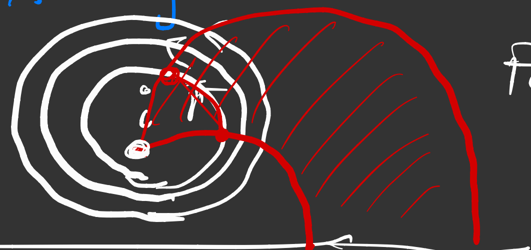
funnel

area() = area() $= \int_0^t \int_0^\infty \frac{dy\,dx}{y^2} = \infty$

area() $= \int_0^t \int_A^\infty \frac{dx\,dy}{y^2} < \infty$

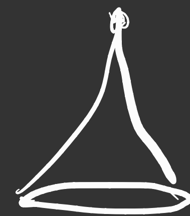
2. $\Gamma = \left\langle \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\rangle$

finite cyclic



conical pt.

For Γ to be discrete $\theta \in \mathbb{Q}\pi$



Prop: If $\mathcal{F}_1, \mathcal{F}_2$ are fundamental domains for a Fuchsian gp. Γ s.t. $\text{area}(\mathcal{F}_1) < \infty$ and both $\text{area}(\partial \mathcal{F}_1) = \text{area}(\partial \mathcal{F}_2) = 0$.

Then $\text{area}(\mathcal{F}_1) = \text{area}(\mathcal{F}_2)$

Proof:

Recall $\mathcal{F}$ is a fund. dom. if

$$\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma \overline{\mathcal{F}} \quad \text{and} \quad \gamma_1 \mathring{\mathcal{F}} \cap \gamma_2 \mathring{\mathcal{F}} = \phi \text{ if } \gamma_1 \neq \gamma_2 \text{ in } \Gamma$$

$$\text{area}(\mathcal{F}_1) \geq \text{area}\Big(\overline{\mathcal{F}}_1 \cap \bigcup_{\gamma \in \Gamma} \gamma \mathring{\mathcal{F}}_2\Big)$$

$$= \text{area}\Big(\bigsqcup_{\gamma \in \Gamma} (\overline{\mathcal{F}}_1 \cap \gamma \mathring{\mathcal{F}}_2)\Big)$$

$$= \sum_{\gamma} \text{area}(\gamma \overline{\mathcal{F}}_1 \cap \mathring{\mathcal{F}}_2)$$

$$\geq \text{area}\Big(\underbrace{\bigcup_{\gamma} \gamma \overline{\mathcal{F}}_1}_{\mathbb{H}} \cap \mathring{\mathcal{F}}_2\Big)$$

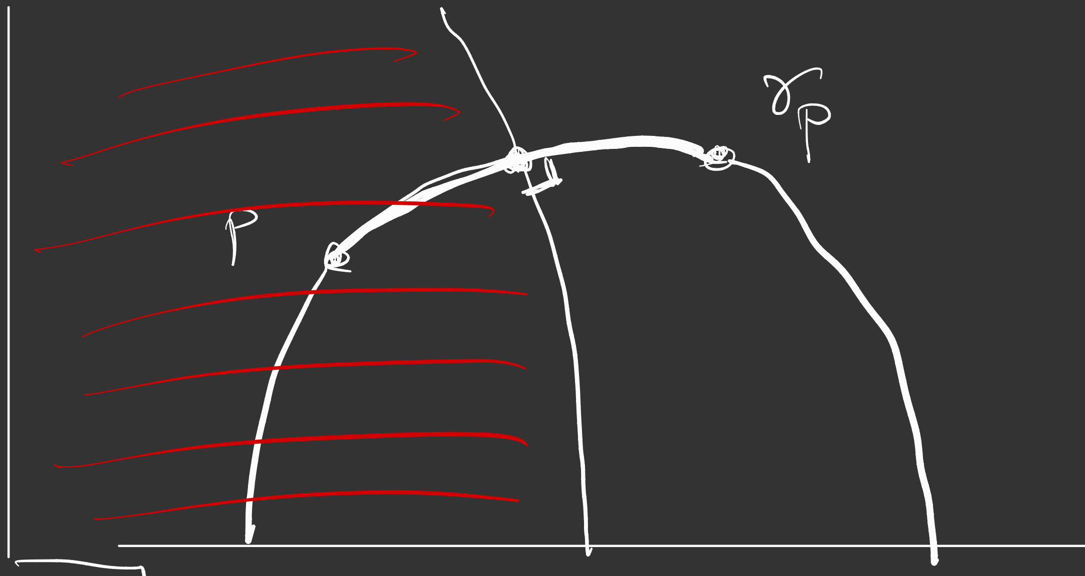
$$= \text{area}(\mathcal{F}_2) \geq \text{area}(\mathcal{F}_1)$$

$\square$

Dirichlet domain for Γ

Fix $p \in \mathbb{H}$ (s.t. p is not fixed by some nontrivial $\gamma \in \Gamma$)

$$\mathcal{D} := \bigcap_{\substack{\gamma \in \Gamma \\ \gamma \neq \pm I}} \{ z \in \mathbb{H} : d(z, p) < d(z, \gamma p) \}$$



$$\{z \in \mathbb{H} : d(z,p) < d(z,\gamma p)\}$$

Prop: $\mathcal{D}$ is a fundamental domain.

Proof:
Let $z \in \mathbb{H}$. We show there exists $\gamma \in \Gamma$, $z_0 \in \mathcal{D}$ s.t. $z = \gamma z_0$, or in other words $\exists\, z_0 \in \mathcal{D} \cap \Gamma z$.

Since Γ is Fuchsian, Γz is discrete, hence $\exists\, z_0 \in \Gamma z$ s.t.

$$d(z_0, p) < d(\gamma z, p) = d(z, \gamma^{-1} p) \quad \forall \gamma \in \Gamma,\ \gamma \neq \pm I$$

$\Rightarrow z_0 \in \mathcal{D}$

Next, let's show that there are no two distinct points in Γz both in $\mathcal{D}$. Let $z_1 \neq z_2 \in \Gamma z$.

In particular, $z_1 = \gamma_0 z_2$ for some $\gamma_0 \in \Gamma$

If $z_1 \in \mathcal{D}$

$$d(z_1, p) < d(z_1, \gamma_0 p) = d(\gamma_0^{-1} z_1, p) = d(z_2, p)$$

If $z_2 \in \mathcal{D}$,

$$d(z_2, p) < d(z_2, \gamma_0^{-1} p) = d(z_1, p) < d(z_2, p)$$

a contradiction

$\blacksquare$

Example: Dirichlet domain for $\Gamma = SL_2\mathbb{Z}$.

$SL_2\mathbb{Z}$ contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$z \mapsto z+1$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -I$

$z \mapsto -\frac{1}{z}$ fixes i

$p = 2i$

(Check that this is not a fixed pt under the action of $SL_2\mathbb{Z}$)

Claim: $\mathcal{T} = \mathcal{D}$

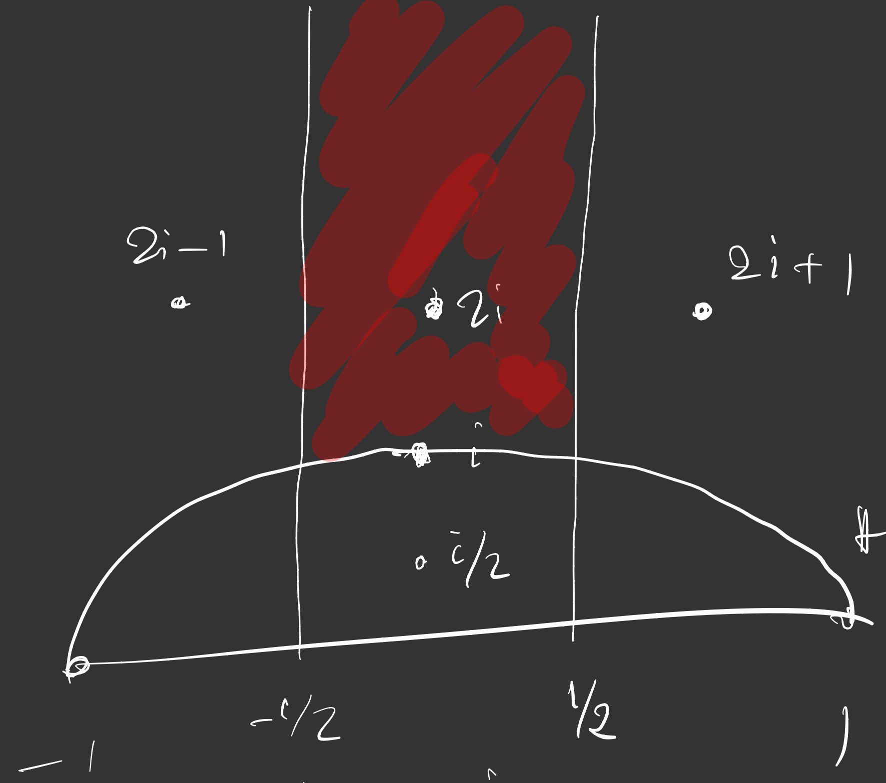
Pf: By construction, $\mathcal{D} \subset \mathcal{T}$.

Suppose that Γz has two distinct points in $\mathcal{T}$.

lemma (exercise):

$\forall z \in \mathcal{T}$, $\forall \gamma \in SL_2\mathbb{Z}$, $\gamma \notin \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$,

$$\underbrace{\mathrm{Im}}_{\text{imaginary part}}(\gamma z) < \mathrm{Im}(z)$$

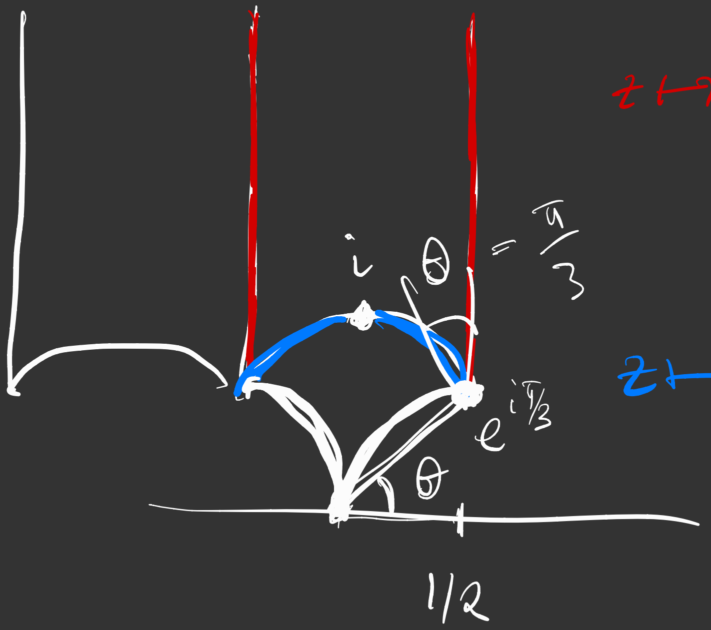


$$2i \mapsto \frac{-1}{2i} = \frac{i}{2}$$

$$\mathcal{T} = \left\{ z \in \mathbb{H} : |\mathrm{Re}\, z| < 1/2,\ |z| > 1 \right\}$$

Suppose $z_1 = \gamma z_2$ are both in $\mathcal{T}$ then

$\mathrm{Im}(z_1) < \mathrm{Im}(z_2) = \mathrm{Im}(\gamma\gamma^{-1} z_2) < \mathrm{Im}(z_1)$ ↯



$z \mapsto z+1$

$z \mapsto -\frac{1}{z}$

cusp:

conical pt. corresponding to i

$$\operatorname{area}(SL_2\mathbb{Z}\backslash\mathbb{H})$$

$$= \pi - 2\frac{\pi}{3} = \frac{\pi}{3}$$

<u>Thm</u> (Siegel)
Let Γ be fuchsian and $\operatorname{area}(\Gamma\backslash\mathbb{H}) < \infty$ then Γ is finitely generated and geometrically finite, i.e. any convex fundamental domain for Γ is a hyperbolic with finitely many sides.

<u>Prop</u>: If $\mathcal{P}$ is a hyperbolic n-gon with inner angles $\alpha_1, \dots, \alpha_n$, then

$$\operatorname{area}(\mathcal{P}) = (n-2)\pi - (\alpha_1 + \dots + \alpha_n)$$

<u>Proof</u>:
Each n-gon can be decomposed into $(n-2)$ adjacent triangles. $\square$

<u>Coro</u>: $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in PSL_2\mathbb{Z}$ generate $PSL_2\mathbb{Z}$.

Some arithmetic constructions of Fuchsian groups (and hyperbolic surfaces)

Fix $N \geq 1$.

$$\Gamma(N) = \ker\left(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})\right)$$

is called a principal congruence group.

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} : \begin{array}{l} a \equiv d \equiv 1 \ (N) \\ b \equiv c \equiv 0 \ (N) \end{array} \right\}$$

def: A subgroup $\Gamma < SL_2\mathbb{Z}$ is called a congruence subgroup if it contains $\Gamma(N) \subset \Gamma$.

Rmk: $\Gamma(1) = SL_2\mathbb{Z}$

Prop: If $N > 1$, then $\Gamma(N)$ contains no elliptic elements.

Proof: $\Gamma(N) < \Gamma(1)$ and the only elliptic elements ($|\text{trace}| < 2$) in $SL_2\mathbb{Z}$ have trace $-1, 0, 1$.

i.e.:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

(up to change of signs)

The lower left entry is always $+1$ or $-1 \not\equiv 0 \ (N)$ if $N \geq 2$. $\square$

On the other hand, $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$