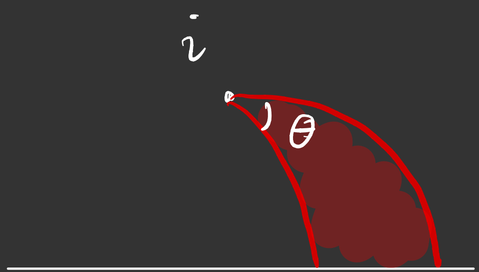
$F \subset \mathbb{H}$ connected set is a <u>fundamental domain</u> for the action of a Fuchsian gp. Γ on $\mathbb{H}$ if

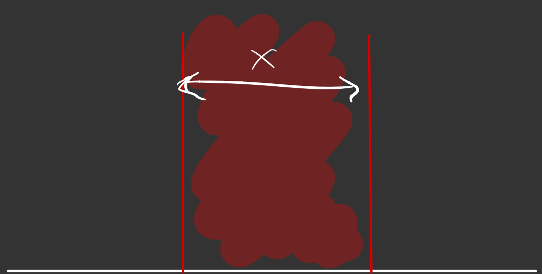
* $\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma \overline{F}$
* $\gamma_1 \mathring{F} \cap \gamma_2 \mathring{F} = \emptyset$ except if $\gamma_1 = \gamma_2$

<u>Examples</u>

$\Gamma = \langle k_\theta \rangle$, $\theta = \frac{\pi}{n}$



$\Gamma = \langle n_x \rangle$, $x > 0$



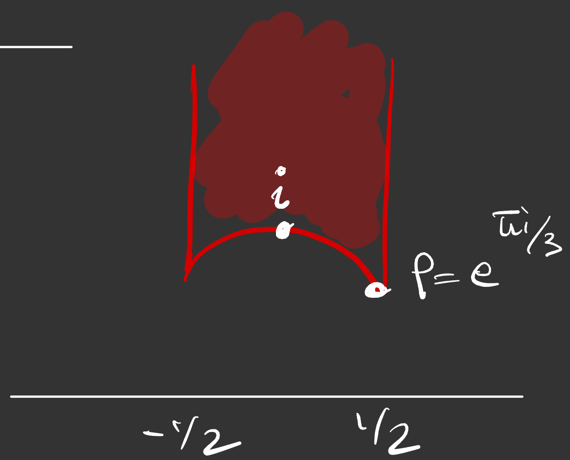
$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\Gamma = SL_2 \mathbb{Z}$

generated by

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

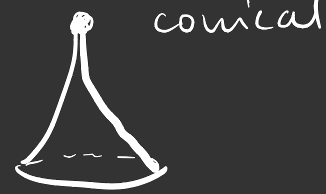
$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

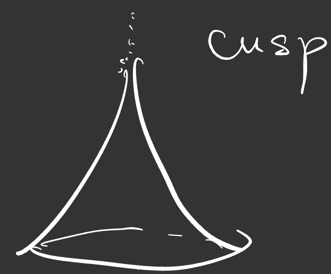


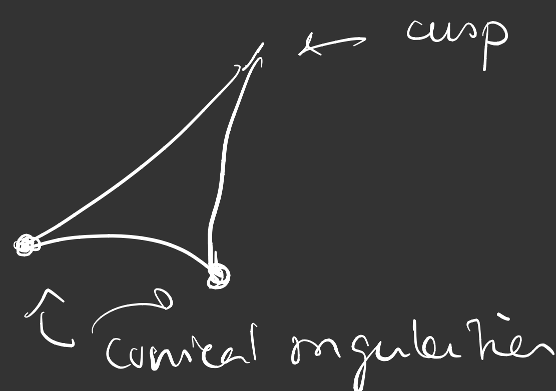
<u>Observations</u>

1. Recall that each $g \in PSL_2 \mathbb{R}$ ($g \neq \pm I$) is either elliptic/parabolic/hyperbolic.

Each comes with specific geometric features:







There is a bijection btw.

$$\left\{\begin{array}{l}\Gamma\text{-orbits of}\\ \text{elliptic fixed}\\ \text{points of } \Gamma\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{conical}\\ \text{singularities}\\ \text{on } \Gamma\backslash\mathbb{H}\end{array}\right\}$$

$$\left\{\begin{array}{l}\Gamma\text{-orbits of}\\ \text{parabolic fixed}\\ \text{pts. of } \Gamma\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{cusp pts}\\ \text{on } \Gamma\backslash\mathbb{H}\end{array}\right\}$$

__Coro:__ $\Gamma\backslash\mathbb{H}$ is compact if and only if Γ contains no parabolic elements.

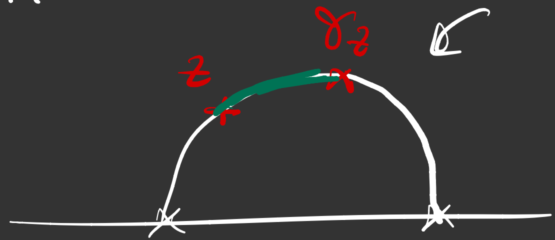
__Prop:__ There is a bijection btw.

$$\left\{\begin{array}{l}\Gamma\text{-conjugacy classes}\\ \text{of hyperbolic}\\ \text{elements in } \Gamma\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{closed}\\ \text{geodesics}\\ \text{on } \Gamma\backslash\mathbb{H}\end{array}\right\}$$

If $\gamma_1, \gamma_2 \in \Gamma$ hyperbolic, then they are in the same Γ-conj. class if $\exists\, \gamma \in \Gamma$ $\gamma_1 = \gamma^{-1}\gamma_2\gamma$

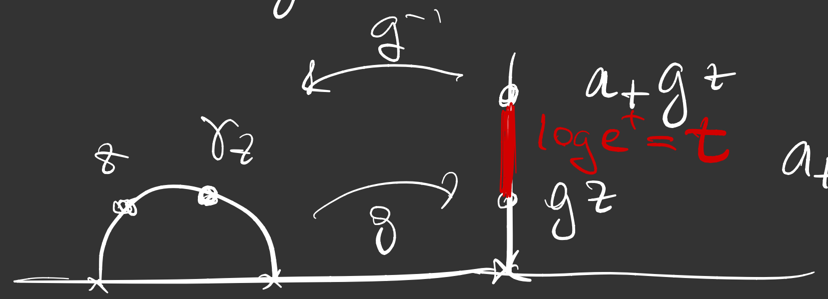
__Proof:__

Let $\gamma \in \Gamma$ be a hyperbolic element $\Rightarrow$ γ has two fixed pts on $\partial\mathbb{H}$



the unique geodesic that joins these pts. is called the axis of γ

__Recall:__ $\exists\, g \in PSL_2\mathbb{R}$



$$a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$$

$$\gamma = g^{-1} a_t g$$

Parametrize $\gamma : \mathbb{R} \to \mathbb{H}$

the green path above $\quad \gamma(0) = z \quad \gamma(t) = \gamma z$

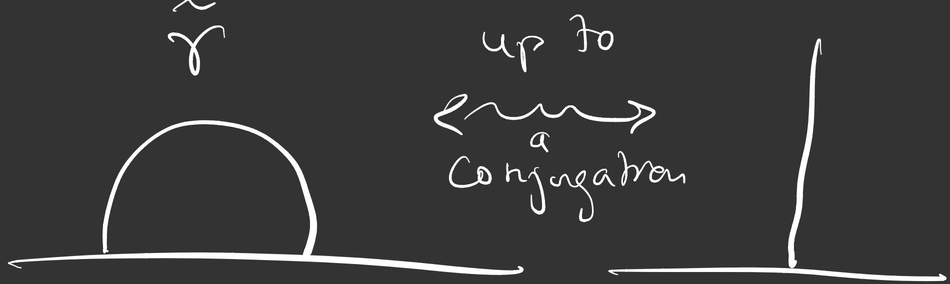
$\gamma(\mathbb{R})$ projected to $\Gamma\backslash\mathbb{H}$ is a closed geodesic

We show that any closed geodesic is obtained in this way. Choose a lift $\tilde{\gamma}$ of this geodesic in $\mathbb{H}$. This is a hyperbolic line.

$G = PSL_2\mathbb{R}$

$$\mathrm{Stab}_G(\tilde{\gamma}) = \{ g \in G : g(\tilde{\gamma}) = \tilde{\gamma} \}$$

$\tilde{\gamma}$



here the stabilizing elements are diagonal matrices

$\Rightarrow$

$$\mathrm{Stab}_G(\tilde{\gamma}) \cong \mathbb{R}$$

$$\mathrm{Stab}_\Gamma(\tilde{\gamma}) = \Gamma \cap \mathrm{Stab}_G(\tilde{\gamma}) \cong \mathbb{Z}$$

Let $\gamma \in \Gamma$ be its generator:

$$\mathrm{Stab}_\Gamma(\tilde{\gamma}) = \langle \gamma \rangle$$

Note: γ is hyperbolic (conjugate to a diagonal element)

What happens if we choose a different lift $\hat{\gamma}$?

Then $\hat{\gamma}$ can be written as $\gamma_0 \tilde{\gamma}$ for some $\gamma_0 \in \Gamma$, so that

$$\mathrm{Stab}_\Gamma(\hat{\gamma}) = \{ \gamma \in \Gamma : (\gamma_0^{-1} \gamma \gamma_0)\tilde{\gamma} = \tilde{\gamma} \}$$
$$= \langle \gamma_0^{-1} \gamma \gamma_0 \rangle$$

$\boxed{\equiv}$

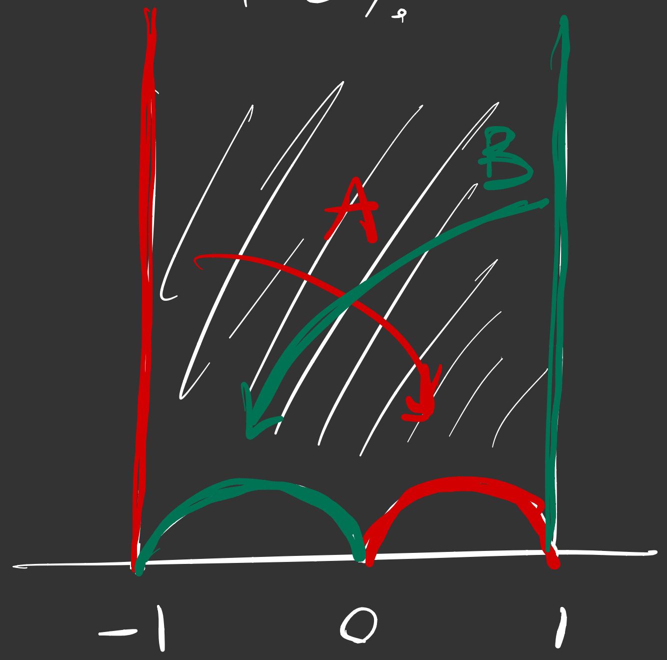
Example: $\Gamma = SL_2\mathbb{Z}$

$\Gamma' = [\Gamma, \Gamma]$ the commutator subgroup of the modular group

$= \{[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} : \gamma_1, \gamma_2 \in \Gamma\}$

is generated by

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}_\infty$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$



Ex: show that this is a fund. domain for Γ'

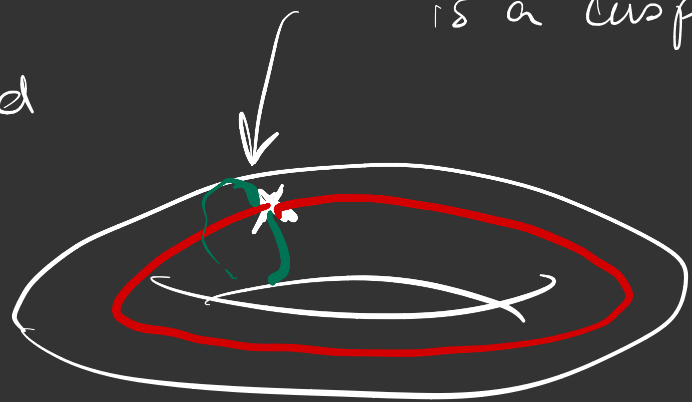
$A(\infty) = 1$
$A(-1) = 0$

$B(\infty) = -1$
$B(1) = 0$

All vertices at ∞ (meaning all vertices of the fund. dom. on $\partial\mathbb{H}$) are in the same Γ'-orbit

once-punctured torus

this puncture is a cusp



Second observation from these few examples: one can read the generators of Γ off the fundamental domain.

<u>Prop</u>: Let $\mathcal{F}$ be a fundamental domain for Γ. Then

$$S = \{\gamma \in \Gamma : \gamma\overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}$$

generates Γ.

<u>Proof</u>:

Let $z \in \mathbb{H}$ w/ $\exists \gamma \in \Gamma$ s.t. $\gamma z \in \overline{\mathcal{F}}$

Suppose $\exists \gamma' \neq \gamma$ in Γ s.t.

$\gamma' z \in \overline{\mathcal{F}}$

Then $\gamma' z \in \overline{\mathcal{F}} \cap \gamma'\gamma^{-1}\overline{\mathcal{F}} \neq \emptyset$

$$\Rightarrow \gamma'\gamma^{-1} \in \Gamma^* = \langle S \rangle$$

So: $\Gamma^*\gamma' = \Gamma^*\gamma$ and we have a function

$$\Phi : \mathbb{H} \longrightarrow \Gamma^* \backslash \Gamma$$
$$z \longmapsto \Gamma^*\gamma$$

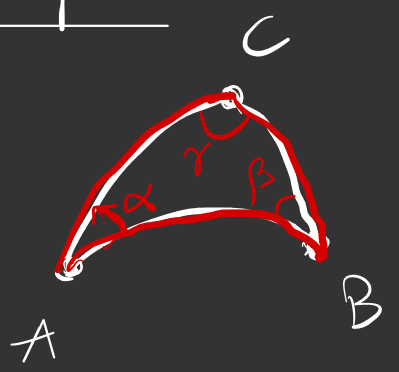
We want to show that Φ is constant. It is enough to check that Φ is loc. constant since $\mathbb{H}$ is connected.

Let K be a compact nbhd. of $z \in \mathbb{H}$.

$$K \subset \bigcup_{i \in I} \gamma_i \overline{\mathcal{F}} \quad \text{with } I \text{ finite}$$

We can take K to be sufficiently small s.t. $z \in \gamma_i \overline{\mathcal{F}}$ for each $i \in I$

<u>Example:</u>



Is there a Fuchsian group with this fundamental domain?

γ_A is the rotation by α about A

γ_B " β " B

γ_C " γ " C

Up to conjugation, each one can be seen as an element of $SO(2)$ $\rightsquigarrow$ $\gamma_A, \gamma_B, \gamma_C \in PSL_2\mathbb{R}$

For $\Gamma \curvearrowright \mathbb{H}$ to be properly discontinuous, we need α, β, γ of the form

Let $z' \neq z$, $z' \in K \Rightarrow z' \in \gamma_j F$ for $j \in I$

and

$$\Phi(z') = \Gamma^* \gamma_j^{-1} = \Phi(z)$$

$$\Rightarrow \Gamma^* = \Gamma$$

$\square$

$$\alpha = \frac{\pi}{a} \quad \beta = \frac{\pi}{b}, \quad \gamma = \frac{\pi}{c}$$

with $2 \leq a, b, c \leq \infty$

Fuchsian groups that arise in this way are called triangle groups.

<u>Rmks</u>:

1. There is a classical thm. of Poincaré (1880s) that formulates under which conditions a

fundamental domain with a prescribed side pairing gives rise to a Fuchsian group

2. $SL_2\mathbb{Z}$ is a triangle group, but most triangle groups are not arithmetic. In fact, there is a classification theorem of Takeuchi (1977) that tells us that up to conjugacy there are only finitely many arithmetic triangle groups

Spectral problem for $M = \Gamma\backslash\mathbb{H}$ compact

Find $\varphi : \mathbb{H} \to \mathbb{C}$ that satisfy

$$\Delta = -y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$$

$$\begin{cases} \Delta\varphi = \lambda\varphi \\ \varphi(\gamma z) = \varphi(z) \quad \forall \gamma \in \Gamma \\ \int_M |\varphi(z)|^2 d\mu(z) = \int_F |\varphi(z)|^2 \frac{dx\,dy}{y^2} \end{cases}$$

F a fundam. domain for Γ

Rmks:

- By the elliptic regularity theorem, any eigenfct. of Δ is automatically smooth.
  admits a countable orthonormal basis

Goal: Find a complete ONB $\{\varphi_k\}_{k\geq 0}$ of Δ-eigenfunctions s.t. any $f \in C^\infty(M)$ has a "spectral expansion"

- $L^2(M)$ is a separable Hilbert space wrt $\langle f, g \rangle = \int_M f(z)\overline{g(z)}\,d\mu(z)$

$$f(z) = \sum_{k\geq 0} \langle f, \varphi_k \rangle \varphi_k(z)$$

Difficulties:

- No explicit solutions to the spectral problem
- Δ is an unbounded operator.