Selberg's insight

spectral theory of Δ on $L^2(M)$ $\rightsquigarrow$ spectral theory of certain Hilbert–Schmidt operators

M compact hyperbolic surface

Plan:

- Review of integral operators
- Selberg's point-pair invariants
- spherical functions (maybe)

An integral operator is a linear operator of the following form:

X loc. cpt. space with a pos. Borel measure

$$T_K : L^2(X) \longrightarrow L^2(X)$$

$$T_K f(x) = \int_X K(x,y) f(y)\,dy$$

K is called a kernel,

$$K \in L^2(X \times X)$$

Rmks:

- $\| T_K f \|_2 \leq \| K \|_2 \| f \|_2$ ($\| K \|_2$: L^2-norm)
- Set $K^*(x,y) = \overline{K(y,x)}$

  Then $\langle T_K f, g \rangle = \langle f, T_{K^*} g \rangle$ for any $f, g \in L^2(X)$

def: $\mathcal{H}$ separable Hilbert space

Let $\{e_i\}_{i \geq 1}$ be an ONB for $\mathcal{H}$

The Hilbert–Schmidt norm of a linear operator $T : \mathcal{H} \to \mathcal{H}$ is

$$\| T \|_{HS}^2 := \sum_{i \geq 1} \| T e_i \|_{\mathcal{H}}^2$$

Rmks:

A linear operator T with $\|T\|_{HS} < \infty$ is called a Hilbert-Schmidt operator.

Thm: (Hilbert-Schmidt)
Every HS operator is compact (i.e. every bdd set is mapped into a compact set).

Thm: (spectral thm. for compact operators)
$\mathcal{H}$ a separable Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ linear compact self-adjoint or normal operator.
Then $\exists$ a complete ONB $\{\varphi_j\}_{j \geq 1}$ of $\mathcal{H}$ composed of T-eigenvectors
$$T\varphi_j = \lambda_j \varphi_j$$
and $\lambda_j \xrightarrow[j \to \infty]{} 0$.

- def. is indep. of choice $\{e_i\}$
- Extends usual def. of Frobenius norm
- $\|T_K\|_{HS} = \|K\|_{L^2}$

Last statement says that each eigenspace E_λ is finite-dimensional

Proof of this last statement

Suppose $\exists$ a subsequence $\{\varphi_j\}$ with $T\varphi_j = \lambda_j \varphi_j$ s.t. $|\lambda_j| > \varepsilon$.

$\mathcal{H}$ separable $\to$ $K \subset \mathcal{H}$ compact iff seq. compact.

Hence T compact implies that since $\{\varphi_j\}$ are bounded, there is a subseq. $(T\varphi_j)$ that converges.

$$\|T\varphi_j - T\varphi_k\|^2 = \|\lambda_j \varphi_j - \lambda_k \varphi_k\|^2$$

def: A point-pair invariant is a function $k: \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C}$

s.t. $k(gz, gw) = k(z, w)$.

$\forall z, w \in \mathbb{H}, g \in \mathrm{Isom}(\mathbb{H})$

In particular, k depends only of $d_{\mathbb{H}}(z, w)$. We obtain a point-pair invariant as

$$k(z,w) := k(d_{\mathbb{H}}(z,w))$$

with k "nice".

For the moment,

"nice" = $k: \mathbb{R} \to \mathbb{R}$ C_c^∞, even

$k(x) = k(-x)$

$T_k : L^2(\mathbb{H}) \longrightarrow L^2(\mathbb{H})$

$$T_k f(z) = \int_{\mathbb{H}} k(z,w) f(w)\, d\mu(w)$$

$= |\lambda_j|^2 + |\lambda_k|^2 > 2\varepsilon^2$

using that φ_j are orthonormal

This is a contradiction $\square$

Again: $\|T_k f\|_2 \le \|k\|_2 \|f\|_2$

$$k^*(x,y) = \overline{k(y,x)} = k(x,y)$$

Rmk: If k is complex-valued, then T_k is normal.

If k is real-valued, then T_k is selfadjoint.

$\rightsquigarrow$ we have a spectral thm. for T_k

def: automorphic kernel

Let k be a "nice" point-pair invariant. Set

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)$$

with Γ Fuchsian group.

Leaving convergence issues aside for the moment, remark:

- K bi-Γ-invariant

$$K(\gamma_1 z, \gamma_2 w) = K(z,w) \quad \forall \gamma_1,\gamma_2\in\Gamma$$

(we have $d_{\mathbb{H}}(z,\gamma w) = d_{\mathbb{H}}(\gamma^{-1}z, w)$)

- K is symmetric:

$$K(z,w) = K(w,z)$$

- $$T_K f(z) = \int_{\mathcal{F}} K(z,w) f(w)\, d\mu(w)$$

$M = \Gamma\backslash\mathbb{H}$ with f. dom. $\mathcal{F}$

if $K\in L^2(M\times M)$, then T_K is compact and self adjoint.

- $$T_K f(z) = \int_{\mathcal{F}} K(z,w) f(w)\, d\mu(w)$$

$$= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma} k(d(z,\gamma w))\, f(w)\, d\mu(w)$$

$$\overset{\text{ahem}}{=} \sum_{\gamma\in\Gamma} \int_{\gamma\mathcal{F}} k(d(z,w))\, f(\gamma^{-1}w)\, d\mu(\gamma^{-1}w)$$

$f(w)\ d\mu(w)$

$$= \int_{\mathbb{H}} k(z,w) f(w)\, d\mu(w)$$

$$= T_k f(z)$$

"folding / unfolding"

Prop: $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$\Delta T_k = T_k \Delta$$

Remark: This implies $\Delta T_K = T_K \Delta$

Proof:

what we want to show ↓

$$\int_{\mathbb{H}} \Delta_z k(z,w) f(w)\, d\mu(w) = \int_{\mathbb{H}} k(z,w)\, \Delta f(w)\, d\mu(w)$$

$$= \int_{\mathbb{H}} \Delta_w k(z,w)\, f(w)\, d\mu(w)$$

This amounts to check that

$$\Delta_z k(z,w) = \Delta_w k(z,w)$$

If we fix $w \in \mathbb{H}$, then we can think of $k(z,w)$ as a function that is radially symmetric about w.

Recall:

$$\cosh \underbrace{d(z,w)}_{r} = 1 + \underbrace{\frac{|z-w|^2}{2yv}}_{2t}$$

$z = x+iy$

$w = u+iv$

Δ in polar coordinates

$(x,y) \rightsquigarrow (r,\theta)$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh(r)}\frac{\partial}{\partial r} + \frac{1}{(2\sinh r)^2}\frac{\partial}{\partial \theta}$$

In polar coordinates about w, we have

$$\Delta = t(t+1)\frac{\partial^2}{\partial t^2} + (2t+1)\frac{\partial}{\partial t} + \frac{1}{16t(t+1)}\frac{\partial^2}{\partial \theta^2}$$

$$\Delta_z k(z,w)$$

$$= \Delta k(t) = t(t+1)k''(t) + (2t+1)k'(t) = \Delta_w k(z,w)$$

Thm: $\exists$ an ONB $\{\varphi_j\}_{j\geq 0}$ in $L^2(M)$ of eigenfunctions of Δ when M is a compact hyperbolic surface.

Let's come back to the question of convergence for

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w).$$

Recall: we are assuming that k is a nice point-pair invariant

$\hookrightarrow$ $k \in C_c^\infty(\mathbb{R})$ even

$$k(z,w) = k(d(z,w))$$

$\Gamma = SL_2\mathbb{Z}$

$$\sum_{\gamma\in\Gamma} k(z,\gamma w)$$

$z = x+iy$

$w = u+iv$

$$\cosh(d(z,w)) = 1 + \frac{|z-w|^2}{2yv}$$

$$= \frac{(x-u)^2}{2yv} + \frac{y}{2v} + \frac{v}{2y}$$

$\to 0$ $\quad \to 1$ if $y=v$

If both $z,w \to \infty$ in $\mathcal{F}$ at roughly the same pace, then $d(z,w)$ stays very small, and within the support of k.

$$\sum_{\gamma\in\Gamma} k(z,\gamma w) \geq \sum_{m\in\mathbb{Z}} k(z, w+m) = \sum_{m\in\mathbb{Z}} U\left(\frac{(x-u-m)^2}{2yv} + \frac{y}{2v} + \frac{v}{2y}\right)$$

$\Gamma = SL_2\mathbb{Z} \supset \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} = \mathrm{Stab}_\Gamma(\infty)$

$U(\cosh(d(z,w))) = k(z,w)$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot z = z+n$

$\mathcal{U}(m) = U($ what's above $)$

Poisson summation = formula

$$\sum_{m\in\mathbb{Z}} \hat{\mathcal{U}}(m) \overset{\text{exercise}}{=} \hat{\mathcal{U}}(0) + O((yv)^{-N}) \quad \forall N \geq 1$$

and $\hat{\mathcal{U}}(0) \sim \sqrt{yv}$ for $\frac{1}{C} v \leq y \leq Cv$

(for some constant $C>0$)

<u>Moral</u>: If M is compact, this issue does not arise and

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w) \text{ is in } L^2(M\times M).$$

Thm: $\exists$ an ONB $\{\varphi_j\}_{j\geq 0}$ in $L^2(M)$ of eigenfunctions of Δ when M is a compact hyperbolic surface.

Δ is well def on $C^\infty(M)$ (dense subsp. of $L^2(M)$)

.... (*)

Proof: Set

$\Sigma = \{$ orthonormal subsets of $L^2(M)$ composed of Δ-eigenfunctions $\}$ ordered by inclusion.

Zorn's lemma: $\exists\, S \in \Sigma$ max.

Set $V = \overline{\operatorname{span}(S)} \subset L^2(M)$ subspace

Let T_k be an invariant integral operator for a nice point-pair invariant k.

We know that $T_k \Delta = \Delta T_k$

In particular, V is both invariant under Δ and T_k, and the same must be true of $V^\perp$ (the orthogonal complement).

We want to show that

$$V^\perp = \{0\}.$$

Since $V^\perp$ is T_k-invariant, the restriction of T_k to $V^\perp$ is again a linear compact selfadjoint operator.

By the spectral thm,

(*) ... where Δ is self-adjoint.

There exist a selfadj. extension of Δ to $L^2(M)$.

This is what the thm. applies to.

$$V^{\perp} = \bigoplus E_\lambda$$ where

E_λ are finite-dim. eigenspaces of T.