Point-pair invariant

$k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$, $k(gz, gw) = k(z, w)$ for all $g \in \mathrm{Isom}(\mathbb{H})$.

$M = \Gamma \backslash \mathbb{H}$ COMPACT hyperbolic surface

$$K(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w)$$

conv. absolutely if $k \in C_c^\infty$. Moreover

- bi-Γ-invariant
- symmetric

$T_K : L^2(M) \longrightarrow L^2(M)$ given by

$$T_K f(z) = \int_{\mathcal{F}} K(z,w) f(w)\, d\mu(w).$$

is

$$\boxed{\begin{array}{l} \bullet \text{ self-adjoint (if } k^* = k) + \text{compact} \\ \bullet \ T_K \Delta = \Delta T_K \end{array}}$$

Spectral thm. for compact operators:
$\mathcal{H}$ separable Hilbert space,
$T : \mathcal{H} \to \mathcal{H}$ linear selfadjoint compact,
Then $\exists$ a complete ONB $\{\varphi_k\}_{k \geq 0}$
s.t. $T\varphi_k = \lambda_k \varphi_k$ with $\lambda_k \xrightarrow{k \to \infty} 0$

Thm (A) M cpt. hyp. surface. $\exists$ a complete ONB $\{\varphi_k\}_{k \geq 0}$ in $L^2(M)$ composed of Δ-eigenfunctions.

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def: A sym. space is a connected Riemannian mfld. for which geodesic inversion at any point is a global isometry.

Rmk: Symmetric spaces of constant curvature are

$$\mathbb{R}^n, \quad \mathbb{S}^n, \quad \mathbb{H}^n$$

Proof of ⊛: We are left to prove that

Claim: If $\mathcal{H}$ is a separable Hilbert space, $T: \mathcal{H} \to \mathcal{H}$ compact, selfadjoint, and commutes with Δ, then $\mathcal{H}$ contains a vector that is an eigenvector of both operators.

Pf of claim:

By the spectral thm for cpt. operators,

$$\mathcal{H} = \bigoplus E_\lambda$$

with E_λ T-eigenspaces that are finite dimensional.

Each E_λ is invariant under both T and Δ:

$v \in E_\lambda \Rightarrow Tv = T(\sum a_i \underbrace{v_i}_{T\text{-eigenv.}})$

$= \lambda \sum a_i v_i \in E_\lambda$

$$\lambda \Delta v = \lambda \Delta (\sum a_i v_i)$$

$$= \lambda \sum a_i \Delta v_i$$

$$= \sum a_i \Delta T v_i$$

$$= \sum a_i T \Delta v_i = T \Delta v$$

$$\Rightarrow \Delta v \in E_\lambda .$$

The restriction of Δ to E_λ is well defined. A linear operator on a finite dimensional linear space has a nonzero eigenvector.

$\square$

$k(z,w)$, a point-pair invariant, with $w \in \mathbb{H}$ fixed, then

$z \mapsto k(z,w)$ is radially symmetric about w.

In fact, the study of point-pair invariants coincides with the study of radial functions on $\mathbb{H}$.

Radializing (symmetrizing $f: \mathbb{H} \to \mathbb{C}$, about $w \in \mathbb{H}$:

Let's assume first $w = i$

$$f_i(z) = \frac{1}{2\pi}\int_0^{2\pi} f(k_\theta z)\, d\theta$$

$$K = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

$$= \mathrm{Stab}_G(i) \qquad G = PSL_2\mathbb{R}$$

Observe:

$\forall k \in K$, $f_i(kz) = f_i(z)$

$$f_i(i) = f(i)$$

More generally, $w = g.i$ for some $g \in PSL_2\mathbb{R}$.

$$\mathrm{Stab}_G(w) = \{h \in G : h.w = w\}$$
$$g^{-1}hg.i = i$$

$$= gKg^{-1}$$

$$f_w(z) = \frac{1}{2\pi}\int_0^{2\pi} f(gk_\theta g^{-1}z)\, d\theta$$

Again:

- f_w is gKg^{-1}-invariant
- $f_w(w) = f(w)$.

def: $f : \mathbb{H} \to \mathbb{C}$ spherical if it is radial about $w \in \mathbb{H}$ and an eigenfunction of Δ

Prop: The space of spherical functions about point $w \in \mathbb{H}$ and eigenvalue λ of Δ is 1-dimensional.

In fact, $\exists!$ $\omega_\lambda(z; w)$ s.t.

$$\Delta_z \, \omega_\lambda(z;w) = \lambda \, \omega_\lambda(z;w)$$

$$\omega_\lambda(w;w) = 1$$

Proof sketch: Let f be a spherical ft. in our space; $\Delta f = \lambda f$. expressed in polar coordinates about w:

$$\cosh(d(z,w)) = 1 + 2u$$

$$(\Delta - \lambda) f(u) = u(u+1) f''(u) + (2u+1) f'(u) - \lambda f(u) = 0$$

$\rightsquigarrow$ has 2 linearly independent (explicit) solutions $F_\lambda(u)$, $G_\lambda(u)$

Fact: As $u \to 0$ (corresponds $z \to w$) then $F_\lambda(u) \to 1$

$$G_\lambda(u) \to \infty$$

$$f(u) = a F_\lambda(u) + b G_\lambda(u)$$

Since $f(0) = f(w)$, we conclude that $b = 0$ and $a = f(w)$. $\square$

Thm: $f:\mathbb{H}\to\mathbb{C}$ $\quad \Delta f=\lambda f$,

k "nice" point-pair invariant,

then $T_k f(z) = \hat{k}(\lambda) f(z)$

with

$$\hat{k}(\lambda) = \int_{\mathbb{H}} k(z_0,w)\,\omega_\lambda(w;z_0)\,d\mu(w)$$

(independent of z_0). (This is called the Selberg transform)

Proof:

We radialize f about $z_0\in\mathbb{H}$.

i.e. we replace f by f_{z_0}.

Observe $\Delta f_{z_0} = \lambda f_{z_0}$.

By preceding proposition,

$$f_{z_0}(z) = a\cdot\omega_\lambda(z;z_0)$$

Using $f_{z_0}(z_0) = f(z_0) = a\cdot\overbrace{\omega_\lambda(z_0;z_0)}^{=1}$

$$T_k f_z(z) = \int_{\mathbb{H}} k(z,w)\,f_z(w)\,d\mu(w)$$

$$= \int_{\mathbb{H}} k(z,w)\,\frac{1}{2\pi}\int_0^{2\pi} f(g k_\theta g^{-1} w)\,d\theta\,d\mu(w)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{H}} \underbrace{k(z,g^{-1}k_{-\theta}g w)}\,f(w)\,d\mu(w)\,d\theta$$

$$k(\underbrace{g k_\theta g^{-1}}_{\in \mathrm{Stab}(z)} z, w) = k(z,w)$$

$$= T_k f(z)$$

$\Rightarrow$

$$T_k f(z) = \int_{\mathbb{H}} k(z,w)\,\underbrace{f_z(w)}_{f(z)\,\omega_\lambda(w;z)}\,d\mu(w)$$

$$= \underbrace{\int_{\mathbb{H}} k(z,w)\,\omega_\lambda(w;z)\,d\mu(w)}_{\text{check that this does not depend on } z}\,f(z)$$

$\square$

Question: If we know that

$T_k f = \mu f$, can we recover what k is?

Rmk: For each eigenvalue λ of Δ, we write

$$\lambda = s(1-s) = \frac{1}{4} + r^2 \quad (r \in \mathbb{C}).$$

We will often write

$$\hat{k}(r) = \hat{k}(\lambda).$$

Prop:
$$\hat{k}(r) = \int_{-\infty}^{\infty} g(u) e^{iru} \, du$$

$$g(u) = \sqrt{2} \int_{|u|}^{\infty} \frac{k(t) \sinh(t)}{\sqrt{\cosh t - \cosh u}} \, dt$$

$$k(t) = -\frac{1}{\pi\sqrt{2}} \int_t^{\infty} \frac{g'(u) \, du}{\sqrt{\cosh u - \cosh t}}$$

(w/o proof).

Application: solving the heat equation on M

We want to find
$$u : M \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$$

s.t.
$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \Delta\right) u(z,t) = 0 \\ u(z, 0) = f(z) \end{cases}$$

for some $f : M \to \mathbb{R}$ continuous

We will try to find a solution of the form
$$u : \mathbb{H} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$$

we will directly assume that $\Delta f = \lambda f$.

By previous theorem:

$$u(z,t) = \hat{p}_t(\lambda) f(z)$$

If u is a solution of (HE) then

$$\frac{\partial u}{\partial t} = \frac{\partial \hat{p}_t}{\partial t} \cdot f$$

$$= -\Delta u = -\hat{p}_t \cdot \Delta f$$

$$= -\hat{p}_t \cdot \lambda \cdot f$$

$$\rightsquigarrow \frac{\partial \hat{p}_t}{\partial t} = -\lambda \cdot \hat{p}_t$$

$$\rightsquigarrow \hat{p}_t(\lambda) = C \cdot e^{-\lambda t}$$

$$u(z,t) = \int_{\mathbb{H}} \underbrace{p_t(z,w)}_{\text{with here a nice (?) point-pair invariant}} f(w)\, d\mu(w)$$

Using now the initial condition

$$u(z,0) = f(z) \Rightarrow \hat{p}_0(\lambda) = 1$$

$$\Rightarrow C = 1$$

Using the previous set of formulas, one can come up with an expression for $p_t(z,w)$.

<u>Fact</u>: The function

$$p_t(z,w) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{d(z,w)}^{\infty} \frac{u e^{-u^2/4t}}{\sqrt{\cosh u - \cosh d(z,w)}}\, du$$

<u>lemma</u>: The series

$$P_t(z,w) = \sum_{\gamma\in\Gamma} p_t(z,\gamma w)$$

converges absolutely.

<u>Proof</u>:

$$|P_t(z,w)| \le \frac{C}{t}\sum_{\gamma\in\Gamma} e^{-d(z,\gamma w)^2/8t}$$

$$= \frac{C}{t}\sum_{n\ge 0} \#\{\gamma\in\Gamma : n \le d(z,\gamma w) < n+1\} \cdot e^{-n^2/8t}$$

What we need is some control on the growth of

$$\#\{\gamma\in\Gamma : d(z,\gamma w) < T\}$$

$$= \#\{z'\in\Gamma w : d(z,z') < T\}$$

$$= O\left(\frac{e^{-d(z,w)^2/8t}}{t}\right)$$

and provides a sol. of (HE) in the form $u(z,t)$.

hyp. disk of radius T

Γw is a discrete orbit in $\mathbb{H}$ (because Γ is Fuchsian)

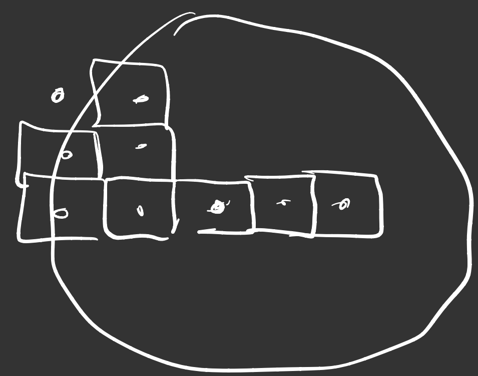
This is the hyperbolic counterpart of the circle problem

$\mathbb{Z}^2$

Count $\#(\mathbb{Z}^2 \cap B_R)$ as $R\to\infty$

Gauss:

$$\#(\mathbb{Z}^2 \cap B_R) \sim \pi R^2$$



Compare the area made out of unit squares centered at $\mathbb{Z}^2$-points with area of B_R.