Exercise set 1

Groups acting on trees

Due by March 5

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by March 5th. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of March 11th.

Trees

These exercises are meant to help you familiarize with some of the basic graph-theoretic notions that we will use throughout the class. Although the group actions we will study are on infinite trees, finite trees will appear often as spanning trees of graphs of groups, and the techniques you need to solve the first exercise will be very handy in that context.

Exercise 1. Let X be a finite graph with n vertices. Consider the three following conditions:

(a) X is connected.

(b) X has no circuit.

(c) X has $(n-1)$ positively oriented edges.

Show that any two of these conditions imply the third.

Hint. For $(a)+(b) \Rightarrow (c)$, start by showing that X has a *leaf* (i.e., a vertex of valency 1), then use induction.

Exercise 2. Let X be a (not necessarily finite) tree, Y a connected graph. Let $\iota : Y \to X$ a morphism such that, for every $y \in Y^0$, the restriction of ι to y and its neighbours is injective: such a morphism is called *locally injective*. Show that ι is actually injective, and deduce that Y is a tree.

Does this imply that being a tree is a local property? That is, given a connected graph, can you check that it is a tree only by looking at each vertex and its neighbours?

Graph automorphisms

These exercises are meant to make you work with the definitions related to graph automorphisms.

Exercise 3. Let X be an infinite tree.

(a) Construct an example where X is such that any automorphism acts without inversion. Can you construct such an example with bounded valency?

(b) Suppose that X is regular. Show that there exists some automorphism that acts with inversion.

(c) What happens if X is finite?

Exercise 4. Let X be a tree and τ an automorphism. Show that τ acts without inversion on the barycentric subdivision $\mathcal{B}(X)$.

Exercise 5. Let X be a tree, let τ, ν be automorphisms and $k \in \mathbb{Z}$ an integer.

(a) Show that $|\nu\tau\nu^{-1}| = |\tau|$.

(b) Assume that τ acts without inversion. Show that $|\tau^k| = |k| \cdot |\tau|$.

(c) Assume that τ acts with inversion. Show that $|\tau| = 1$.

Cayley graphs

The goal of these exercises is to have you draw some Cayley graphs. Don't spend so much time in trying to rigorously prove things, the main objective here is to get familiar with the definition.

Exercise 6. Draw the Cayley graph of the following pairs (G, S), where G is a group and S is a prescribed generating set. In the following items $\overline{k}$ denotes the reduction of k modulo n.

(a) $(S_3, \{(12), (123)\})$.

(b) $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \{(\overline{1}, \overline{0}), (\overline{0}, \overline{1})\})$. Think about the difference between (a) and (b).

(c) $(\mathbb{Z}, S)$, where $S = \{1\}$ and where $S = \{2, 3\}$.

(d) $(\mathbb{Z}^2, S)$, where $S = \{(1, 0), (0, 1)\}$ and where $S = \{(1, 0), (1, 1)\}$.

(e) $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \{(\overline{1}, \overline{0}), (\overline{0}, \overline{1})\})$.

(f) $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}, \{(\overline{1}, 0), (\overline{0}, 1)\})$.

Exercise 7. For each of the following properties of a pair (G, S), decide whether you can detect it by looking at the Cayley graph. What do you need to verify it? Is the underlying unoriented graph enough? Do you also need orientations? Do you also need labels?

(a) S contains the identity.

(b) G contains an element of order 2.

(c) S contains an element of order 2.

(d) S contains an element of finite order.

(e) S contains two elements that commute.

(f) S is not minimal: there exists a proper subset $S' \subset S$ that generates G.