Exercise set 2

Groups acting on trees

Due by March 19

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by March 19th. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of March 25th.

Free groups

Let X be a set and let $F[X]$ be a free group with basis X. We will identify $F[X]$ with the free group constructed in Exercise Class 1 (or in Bogopolsky): the set of equivalence classes of words on $X \cup X^{-1}$, with multiplication given by concatenation. Recall that every element is represented by a unique reduced word: for $g \in F[X]$ we denote by $|g|$ the length of the reduced representative.

The exercises in this and the next section are about algebraic properties of free groups. Since two free groups on X are isomorphic, and free groups are characterized by their universal property, below you can either use the definition, work with the explicit construction of $F[X]$, or use the universal property. The goal is to get used to play with both combinatorics and algebra to understand the free group.

Exercise 1. Show that for any non-identity element $g \in F[X]$, the sequence $\{|g^n|\}_{n\geq 1}$ is strictly increasing. Deduce that non-trivial free groups are torsion-free.

Exercise 2. Let $x \in X$. Show that the centralizer in $F[X]$ of x is the group generated by x. Deduce that free groups of rank at least 2 have trivial center.

Exercise 3. Show that the abelianization of $F[X]$ is isomorphic to $\mathbb{Z}[X]$: the free abelian group with basis X. Use this to give another proof that any two bases of $F[X]$ have the same cardinality.

Hint. Recall from Algebra 1 that free abelian groups are also characterized by a universal property.

Free subgroups

The following exercises are about free subgroups of certain groups. You will see explicit examples later on, once you introduce the ping-pong lemma, but for the moment we can give a recipe to construct free subgroups from free quotients.

Exercise 4. Let G be a group, F be a free group and $\varphi : G \to F$ a surjective homomorphism. Show that there exists a homomorphic section, that is, a homomorphism $\sigma : F \to G$ such that $\varphi\sigma = id_F$. Deduce that G contains a subgroup isomorphic to F.

The following exercise has as a main goal to prove that a free group of finite rank contains free subgroups of finite index and arbitrary rank. This is completely different to free abelian groups, or vector spaces! It is the hardest one of this exercise sheet, so feel free to just hand in a solution to (a) or (b) if you're not able to do both.

Exercise 5. Let $X = \{a, b\}$, and let $X_k = \{b, aba^{-1}, a^2ba^{-2}, \ldots, a^{k-1}ba^{1-k}, a^k\}$ for $k > 1$.

(a) Show that X_k generates the kernel of the homomorphism $F[X] \to \mathbb{Z}/k\mathbb{Z}$ defined on the basis by $a \mapsto 1, b \mapsto 0$.

(b) Show that the subgroup generated by X_k is free with basis X_k.

Deduce that a free group of finite rank contains free subgroups of finite index and arbitrarily large finite rank, and these subgroups can be chosen to be normal. Can you find a free subgroup of infinite rank?

Presentations

The goal of these exercise is to make you work with group presentations by explicitly constructing a few simple ones.

Exercise 6. Let G be a finite group of order n. Explain how to construct a finite presentation of G. How many generators does it have in terms of n? How many relators? What is the length of the relators?

Apply this to the group $\mathbb{Z}/n\mathbb{Z}$, and compare the presentation you obtain to the standard presentation $\langle x \mid x^n \rangle$.

Exercise 7. Let $G = \langle S_G \mid R_G \rangle$ and $H = \langle S_H \mid R_H \rangle$ be two groups defined by finite presentations. For each of the constructions below, explain how to construct a finite presentation using the given ones of G and H.

(a) The direct product $G \times H$. Apply this to the group $\mathbb{Z}^2$.

(b) More generally, a semidirect product $G \rtimes_\varphi H$, where $\varphi : H \to Aut(G)$ is a homomorphism. Apply this to the dihedral group D_n.

Remark. Recall that $G \rtimes_\varphi H$ is the group with underlying set the Cartesian product $G \times H$, and with composition defined by

$$(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)(g_2), h_1h_2).$$

When $H = \mathbb{Z}$, the semidirect product is a special case of a more general construction that you will see plenty of in the class: the *HNN extension.*