Exercise set 3

Groups acting on trees

Due by April 9

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by April 9th. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of April 15th.

Ping-pong

The following exercise is a typical application of the ping-pong lemma. The end goal is to show that $\mathrm{SL}_2(\mathbb{Z})$ is *virtually free*, that is, it contains a free subgroup of finite index.

Exercise 1. Let $G \leq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \quad y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

(a) Use the ping-pong lemma on the action of G on $\mathbb{Z}^2$ to show that G is free with basis $\{x, y\}$.

(b) Show that G contains the modulo 4 congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, namely $\mathrm{SL}_2(\mathbb{Z})_4 := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \mod 4\}$.

(c) Deduce that $\mathrm{SL}_2(\mathbb{Z})$ is virtually free. Is $\mathrm{SL}_2(\mathbb{Z})$ free?

Hint. For (a), think about what happens to the absolute values of the entries of a point in $\mathbb{Z}^2$ after applying x or y.

Free products

The next exercise shows that free products contain large free subgroups.

Exercise 2. Let G and H be groups. Consider the natural homomorphism $\varphi : G * H \to G \times H$, that is, the unique homomorphism such that $\varphi|_G : G \to G \times H : g \to (g, 1)$ and $\varphi|_H : H \to G \times H : h \mapsto (1, h)$.

(a) Show that the kernel of φ is free with basis $\{[g, h] : g \in G \setminus \{1\}, h \in H \setminus \{1\}\}$.

(b) Deduce that the commutator subgroup of F_2 is free of infinite rank (compare with the last question of Exercise 5 in Exercise set 2).

(c) Deduce that a free product of two finite groups is virtually free. Can it be free?

(d) Use (c) to give another proof that $\mathrm{SL}_2(\mathbb{Z})$ is virtually free.

The next exercise is about expressing a group as a free product, even though it does not look like a free product at first sight. This will be a recurring theme throughout the lecture, as group actions on trees allow to obtain similar decompositions.

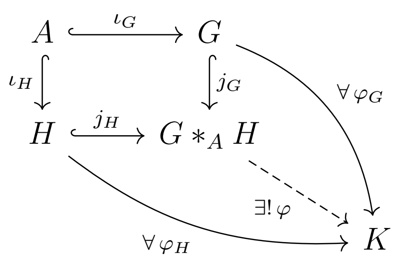
Exercise 3. Consider the semidirect product $G := F_2 \rtimes \mathbb{Z}/2\mathbb{Z} = \langle a, b, t \mid tat^{-1} = b, t^2 = 1 \rangle$. Show that $G \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Amalgamated products

The next exercise is about the universal property of amalgamated products that you saw in class. If you find it too hard, you are encouraged to first try the case of free products, which corresponds to the case in which $A = \{1\}$.

Exercise 4. Let A, G, H be groups, and $\iota_G : A \to G, \iota_H : A \to H$ injective homomorphisms. Let $G *_A H$ be the corresponding amalgamated product, with the canonical injective homomorphisms $j_G : G \to G *_A H$ and $j_H : H \to G *_A H$.

(a) Show that $G *_A H$ enjoys the following universal property. For every group K and every pair of homomorphisms $\varphi_G : G \to K, \varphi_H : H \to K$ such that $\varphi_G \circ \iota_G = \varphi_H \circ \iota_H : A \to K$, there exists a unique homomorphism $\varphi : G *_A H \to K$ such that $\varphi_G = \varphi \circ j_G$ and $\varphi_H = \varphi \circ j_H$.



(b) Show that this universal property characterizes $G *_A H$. That is, show that if L is a group with homomorphisms $j_G : G \to L, j_H : H \to L$ such that $j_G \circ \iota_G = j_H \circ \iota_H : A \to L$, and L has the universal property above, then there exists a canonical isomorphism $L \cong G *_A H$.

The goal of the next exercise is to practice working with normal forms on the *torus knot group*, a very cool and important group from knot theory. When m and n are coprime, this is the fundamental group of $\mathbb{R}^3 \setminus K$, where K is the (m, n)-*torus knot*. You are encouraged to go look for pictures online!

Exercise 5. Let $m, n \in \mathbb{Z} \setminus \{0\}$, and consider the *torus knot group* $K_{m,n} := \langle a, b \mid a^m = b^n \rangle$.

(a) Show that $K_{m,n} \cong K_{-m,n} \cong K_{m,-n} \cong K_{n,m}$.

(b) Express $K_{m,n}$ as an amalgamated free product, find transversals, and describe the normal forms with respect to these.

(c) Prove that the amalgamated subgroup $\langle a^m \rangle = \langle b^n \rangle$ is contained in the center of $K_{m,n}$.

(d) Let $m = 7$ and $n = -6$. Find the normal forms of the following elements:

- $a^{-3}b^2(ab)^3b^{-5}$.
- $b^{11}a^{23}b^{-1}ab^{-11}$.
- $(ab)^{100}$.