Exercise set 4

Groups acting on trees

Due by April 23

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by April 23rd. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of April 29th.

The Baumslag–Solitar groups

This section contains exercises about the Baumslag–Solitar groups, the HNN analogue of the torus knot groups that we saw in the previous exercise sheet. First, let us get acquainted with the definition and practice with normal forms.

Exercise 1. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the *Baumslag–Solitar group*

$$BS(m,n) := \langle a, t \mid t^{-1}a^m t = a^n \rangle.$$

(a) Express $BS(m,n)$ as an HNN-extension, find transversals and describe the normal form with respect to these.

(b) Let $m = 7$ and $n = -6$. Find the normal form of the following elements.

- $tata^{-1}ta^2ta^{-2}ta^3ta^{-3}$.
- $t^{-1}at^{-1}a^{-1}t^{-1}a^2t^{-1}a^{-2}t^{-1}a^3t^{-1}a^{-3}$.
- $ta^{100}ta^{-100}t$.

A special subclass of Baumslag–Solitar groups admits a nicer description.

Exercise 2. Show that $BS(1,n) \cong \mathbb{Z}\left[\frac{1}{n}\right] \rtimes_\varphi \mathbb{Z}$, where $\varphi : \mathbb{Z} \to Aut(\mathbb{Z}\left[\frac{1}{n}\right])$ is defined by $\varphi(k)(x) = n^k \cdot x$. The group $BS(1,-1)$ is commonly known as the *Klein bottle group* (why?) and the group $BS(1,1)$ is commonly known as...?

Isomorphisms between Baumslag–Solitar groups are more restrictive than between torus knot groups (compare with the first question of Exercise 5 in Exercise set 3). The following proposition says exactly when two Baumslag–Solitar groups are isomorphic, and you are asked to prove part of it.

Proposition (Moldavanskii, 1991)**.** Let $m, n, m', n' \in \mathbb{Z} \setminus \{0\}$. Then $BS(m,n) \cong BS(m',n')$ if and only if $(m',n') \in \{(m,n), (-m,-n), (n,m), (-n,-m)\}$.

Exercise 3. Prove $\Leftarrow$ of the proposition above. Prove $\Rightarrow$ in case $(m,n) = (1,1)$.

In the next exercise we will see the reason why these groups were introduced by Baumslag and Solitar in 1962: to disprove several conjectures and false claims of the time. We will look at the ones on Hopfian groups.

Definition. A group G is *Hopfian* if every surjective homomorphism $G \to G$ is injective.

Most of the finitely generated groups you encounter in daily life are Hopfian. In the 50's Higman (the H of HNN) claimed that a finitely presented group defined by a single relation is Hopfian; and B.H. Neumann (an N of HNN) asked whether there exists a 2-generated finitely presented non-Hopfian group. Now $BS(2,3)$ is not Hopfian:

Exercise 4. Let again $BS(2,3) = \langle a, t \mid t^{-1}a^2t = a^3 \rangle$. Set $\eta(a) = a^2$ and $\eta(t) = t$.

(a) Show that η defines a homomorphism $BS(2,3) \to BS(2,3)$.

(b) Find an element that maps to a, and deduce that η is surjective.

(c) Use this to find a non-trivial element in the kernel, and deduce that η is not injective.

Remark. This proof generalizes to $BS(m,n)$ as long as there exists a prime p dividing m but not n. It turns out that this hypothesis is necessary: if m and n have the same set of prime divisors, then $BS(m,n)$ is Hopfian (but the proof is more involved).

The Higman group

In this exercise you will apply what you learned about amalgamated products to get to know the *Higman group*: a finitely presented infinite group without non-trivial finite quotients. Most of the finitely generated groups you encounter in daily life have plenty of finite quotients, and in fact the existence of such a group was open until its construction in 1951.

In the first exercise you are asked to analyze the structure of the Higman group in terms of amalgamated products. In the second one you are asked to show that it has no non-trivial finite quotients.

Exercise 5. Consider the *Higman group*

$$G := \langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle.$$

Denote by $G_{ab} := \langle a, b \mid bab^{-1} = a^2 \rangle \cong \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}$, and similarly G_{bc}, G_{cd}, G_{da}.

(a) Let $G_{abc} := G_{ab} *_{\langle b \rangle} G_{bc}$. Show that a and c generate freely a subgroup $F \leq G_{abc}$.

(b) Define similarly G_{cda}. Show that $G \cong G_{abc} *_F G_{cda}$ and deduce that G is infinite.

Exercise 6. Use the following two steps to show that all finite quotients of G are trivial.

(a) Show that G admits an automorphism permuting cyclically a, b, c, d. Deduce that if G has a non-trivial finite quotient, then it also has one in which a, b, c, d all have the same order.

(b) Show that in a finite group, if g is conjugate to g^2 by an element of the same order, then $g = 1$.