Exercise set 5

Groups acting on trees

Due by May 7

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by May 7th. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of May 13th.

From amalgams to trees

After working combinatorially on free products with amalgamations and HNN-extensions in the past two exercise sets, let us get familiar with the correspondence with trees.

Exercise 1. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the torus knot group $K_{m,n} = \langle a, b \mid a^m = b^n \rangle$ (cf. Exercise set 3). Describe the tree obtained via Theorem 6.12: define it, draw a picture, and compute the degrees of its vertices.

Exercise 2. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the Baumslag–Solitar group $BS(m,n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle$ (cf. Exercise set 4). Describe the tree obtained via Theorem 7.14: define it, draw a picture, and compute the degrees of its vertices.

Uniqueness of the Klein bottle group

Recall the following proposition that was already mentioned in Exercise set 4:

Proposition (Moldavanskii, 1991)**.** Let $m, n, m', n' \in \mathbb{Z} \setminus \{0\}$. Then $BS(m,n) \cong BS(m', n')$ if and only if $(m', n') \in \{(m,n), (-m,-n), (n,m), (-n,-m)\}$.

We proved $\Leftarrow$ in general, and $\Rightarrow$ in case $(m,n) = (1,1)$, that is $\mathbb{Z}^2$ is unique among Baumslag–Solitar groups. Now we will prove $\Rightarrow$ in case $(m,n) = (1,-1)$, that is the Klein bottle group is unique among Baumslag–Solitar groups, using actions on trees. The starting point is given by the following two exercises, which are interesting results by themselves.

Exercise 3. Let σ, τ be commuting automorphisms of a tree X acting without inversion.

(a) Show that σ preserves $\mathring{\tau}$ or $\vec{\tau}$ (depending on whether τ is a rotation or a translation).

(b) Deduce that $\langle \sigma, \tau \rangle$ has a global fixed point or preserves a line in X.

Exercise 4. Let A be a finitely generated abelian group acting without inversion on a tree X.

(a) Show that A has a global fixed point or preserves a line in X.

(b) Deduce that if the quotient $A \setminus X$ is finite, then either X is finite or there exists a line L (respectively, a point p) and an integer $d \geq 1$ such that every vertex in X is at distance at most d from a vertex in L (respectively, from p).

(c) Generalize the previous item to finitely generated virtually abelian groups (i.e., groups with an abelian subgroup of finite index).

We are ready to prove:

Exercise 5. Show that the Klein bottle group is virtually abelian. Then use Exercises 2 and 4 to prove $\Rightarrow$ of Moldavanskii's Proposition in case $(m, n) = (1, -1)$.

Generalized Baumslag–Solitar groups

Both torus knot and Baumslag–Solitar groups fall into the framework of *Generalized Baumslag–Solitar groups.*

Definition. Let Y be a finite connected graph with a spanning tree T, and a *weight function*, i.e. a map $w : E(Y) \to \mathbb{Z} \setminus \{0\}$. Let G be a group generated by elements $\{g_v : v \in V(Y),\, t_e : e \in E(Y) \setminus E(T)\}$ subject to the relations $\{g_{\alpha(e)}^{w(\overline{e})} = g_{\omega(e)}^{w(e)} : e \in E(T)\}, \{t_e t_{\overline{e}} = 1, t_e g_{\alpha(e)}^{w(\overline{e})} t_e^{-1} = g_{\omega(e)}^{w(e)} : e \in E(Y) \setminus E(T)\}$. Then G is called the *GBS group* corresponding to the weighted graph (Y, w).

Exercise 6. Show that GBS groups act on trees without inversion with finite quotient and all vertex and edge stabilizers infinite cyclic.

Remark. Next week we will see that the converse also holds: all groups admitting such actions are GBS groups.

Exercise 7. Write down presentations of the GBS groups corresponding to the following weighted graphs. Below, an orientation of each edge is chosen, and the oriented edge e is labeled by $(w(\overline{e}), w(e))$.