Exercise set 6

Groups acting on trees

Due by May 21

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by May 21st. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. Solutions will be presented during the exercise class of May 27th.

Bass–Serre trees

If G is the fundamental group of a graph of groups, then the tree constructed in Theorem 8.10 is called the *Bass–Serre tree of* G. With these exercises, we get familiar with the construction.

Exercise 1. Let G be a free product with amalgamation (respectively, an HNN extension). Express G as the fundamental group of a graph of groups, and show that its Bass–Serre tree coincides with the tree obtained via Theorem 6.12 (respectively, Theorem 7.14).

Exercise 2. Consider the two GBS groups from Exercise 7 in Exercise set 5. Describe their Bass–Serre trees: define them, draw pictures, and compute the degrees of their vertices.

Finite vertex groups

The goal of this section is to prove the following application of the Fundamental Theorem of Bass–Serre theory, which vastly generalizes Exercise 2 in Exercise set 3.

Proposition (Bass–Serre)**.** Let G be the fundamental group of a finite connected graph of groups with finite vertex groups. Then G is virtually free.

Remark. Time permitting, by the end of the lecture we will also prove the converse.

The following exercise, which is where Bass–Serre theory comes into play, tells you where to find the free subgroup.

Exercise 3. Let H be a normal subgroup of G that intersects trivially each vertex group. Show that H is free.

So the goal now is to find such an H which is moreover of finite index. To do this, we will construct a special homomorphism of G to a finite symmetric group, and H will be the kernel. We first need a basic fact from finite group theory.

Exercise 4. Let F be a finite group and $n = d \cdot |F|$ for some $d \geq 1$. We can embed F into S_n as follows: choose injective maps $\iota_1, \dots, \iota_d : F \to \{1, \dots, n\}$ with disjoint image, and let F act on $\iota_i(F)$ by $g \cdot \iota_i(x) = \iota_i(gx)$. We call such an embedding *standard*.

(a) Show that there is a correspondence between standard embeddings of F into S_n and free actions of F on $\{1, \dots, n\}$.

(b) Deduce that any two standard embeddings of F into S_n are conjugate.

(c) Deduce that any standard embedding of a subgroup of F into S_n can be extended to a standard embedding of F into S_n.

Exercise 5. Let G be as in the proposition, and let n be a multiple of the order of all vertex groups. Use Exercise 4 to define a homomorphism $G \to S_n$ such that the restriction to each vertex group is a standard embedding, and show that the kernel satisfies the hypotheses of Exercise 3.

Hint. Start by assuming that the underlying graph is a tree: for this you only need (c).

The hypothesis that the underlying graph is finite is necessary, as the next exercise shows.

Exercise 6. Let G be a *locally finite group*, that is, a group such that every finitely generated subgroup is finite. Suppose moreover that G is countably infinite. Show that G is the fundamental group of a connected graph of groups with finite vertex groups, but it is not virtually free.

Examples of such groups include the group S_∞ of finitely supported permutations of $\mathbb{N}$, and the rational subgroup $\mathbb{Q}/\mathbb{Z}$ of the circle group.

Invariant free factors of free groups

In this section we apply the Kurosh Subgroup Theorem to prove part of the following proposition.

Proposition (Scott)**.** Let F be a finitely generated free group, α an automorphism of F and H a finitely generated subgroup of F. If $\alpha(H) \subseteq H$, then $\alpha(H) = H$.

We prove the proposition in case H is a *free factor* of F: namely, there exists another subgroup K such that $F = H * K$ (actually this special case is a step towards the proof of the general case, which uses some more advanced tools).

Exercise 7. Let F, α, H be as in the proposition, and assume that H is a free factor of F.

(a) Suppose that $H' \leq H$ is another free factor of F. Show that H' is a free factor of H.

(b) Apply this to $H' = \alpha(H)$, and compare the ranks to conclude.