Exercise set 7

Groups acting on trees

Due by June 4

You can upload your solution to one exercise to `sam-up.math.ethz.ch` by June 4th. You are encouraged to work in pairs, which allows you to work together on two problems and get both solutions corrected. This is bonus material, so solutions will be *not* be presented.

Number of ends

The goal of this section is to prove the following theorem:

Theorem (Freudenthal)**.** Let G be a finitely generated group. Then $e(G) \in \{0, 1, 2, \infty\}$, where $e(G)$ denotes the number of ends of G.

Recall that in class we saw example of all four cases. We also saw examples of graphs whose number of ends does not fit in this range, so the theorem should use in an important way the group structure.

Throughout, G is a finitely generated group. If S is a finite generating set, we denote by $\Gamma(G, S)$ the corresponding Cayley graph, endowed with the path distance d, and by $E(G, S)$ the set of ends of $\Gamma(G, S)$. We admit that the number of ends of G is well-defined; namely, any two Cayley graphs of G have the same number of ends.

Exercise 1. (a) Show that G admits a natural action on the set of rays in $\Gamma(G, S)$, and that this action preserves equivalence. Therefore G acts on $E(G, S)$.

(b) Deduce that if $e(G) < \infty$, then there exists a finite-index subgroup of G that acts trivially on $E(G, S)$, i.e. it fixes every end.

The proof goes by contradiction. Suppose that $3 \leq e(G) < \infty$. Let H be the finite-index subgroup given by the previous exercise.

Exercise 2. Show that there exist three rays $r_k = (e_i^k)_{i \geq 0}$ $(k = 0, 1, 2)$ with the following properties:

- Each r_k is based at the origin: $\alpha(e_0^k) = 1_G$.
- Each r_k is geodesic: $d(\alpha(e_i^k), 1_G) = i$.
- There are infinitely many vertices of r_0 representing elements of H.
- There exists $l \geq 0$ such that $r_k \setminus B_l$ lie in three distinct connected components of $\Gamma(G, S) \setminus B_l$.

Here B_l denotes the open ball of radius l in $\Gamma(G,S)$ with respect to the path distance d. We will now provide two contradictory estimates for the distance between a vertex of r_1 and a vertex of r_2.

Exercise 3. Let t_k be a vertex in r_k $(k = 1, 2)$ with $d(t_k, 1_G) > 2l$. Prove that $d(t_1, t_2) > 2l$.

The other estimate requires some more work.

Exercise 4. Let h be an element of H which belongs to r_0 and such that $d(h, 1_G) > 3l$.

(a) Show that such an element exists.

(b) Show that there exists a vertex t_k in r_k $(k = 1, 2)$ with distance larger than $2l$ from the origin, such that $h \cdot t_k \in B_l$.

(c) Estimate $d(t_1, t_2)$ and conclude.

Virtually free groups

In this section we use Stalling's Theorem to give his proof of the following result:

Theorem (Serre's Conjecture – Wall's Theorem)**.** Let G be a finitely generated virtually free group. If G is torsion-free, then G is free.

This is a special case of the more general fact that finitely generated virtually free groups are fundamental groups of finite connected graphs of groups with finite vertex groups (we proved the converse in the previous exercise sheet). The following exercise illustrates the strength of this theorem:

Exercise 5. Provide an example of a group which is finitely generated, virtually free abelian, torsion free, but not abelian.

The proof of Serre's conjecture uses an induction argument. It applies the following two ingredients:

Proposition. Let G be a finitely generated group, $H \leq G$ a finite-index subgroup. Then H is finitely generated and $e(H) = e(G)$.

Recall that the *rank* of G is the minimal number of generators of G.

Theorem (Grushko)**.** Let G_1, G_2 be groups of rank $n_1, n_2 < \infty$. Then $G_1 * G_2$ has rank $n_1 + n_2$.

Unfortunately the proofs of these results would take us too much time, so we will instead admit them and focus on the application of Stallings's Theorem. So that you know: the first is not that hard to prove (but it does take some time), and the second can be proven using Bass–Serre theory.

We start with the base-case of the induction:

Exercise 6. Let G be virtually cyclic and torsion-free. Show that G is cyclic.

Hint. Such a group cannot contain a non-abelian free subgroup.

Using this for the base case, and Stallings's and Grushko's Theorems for the induction step:

Exercise 7. Prove Serre's Conjecture.